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NON-CONSERVATIVE MAXIMUM FLOW MINIMUM COST SOLUTION IN UNCERTAIN NETWORK

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Abstract

Uncertainty theory examines the behaviors of uncertain parameters on networks. We consider uncertain and predetermined capacities, respectively, on arcs and intermediate vertices of an uncertain network. An objective is to find the non-conservative flow that is maximum at the destination and also at intermediate nodes with minimum cost. This goal is achieved by determining minimum cost paths which send the maximum flow to the sink. In this paper, we solve this problem by incorporating the idea of uncertainty theory. We define this problem, give its mathematical model and present efficient algorithms to solve our newly formulated problem. The illustrated example verifies more efficiency of our approach since it increases the flow values by 19% to 44% for different confidence levels than that of the classical solutions. The relation of confidence level with the maximum flow value and the minimum cost are also observed.

Index Terms: Maximum Flow Model, Intermediate Storage, Cost Minimization Model, Network Simplex Method, Uncertainty Theory.

1. INTRODUCTION

The vertex and arc parameters in majority of traditional network optimization problems have deterministic values. Numerous scholars applied their findings in these deterministic networks to both flow maximization and cost minimization problems. Flash back on maximum flow problem (MFP) and minimum cost problem (MCP) are mentioned below. Now we consider a simple example to highlight how non-conservative flow optimizes the flow value. Only 15.5 units of the maximum amount of commodities can be moved to Vertices 3 and 4, if the flow directions are as shown in Figure 1. This is because the capacity of the arcs are given beside them. However, if we have the capability of storing the commodities in Vertex 2, we can remove 20.5 units of commodities from Vertex 1, which is higher than the prior value. The flow of issues like evacuation, network communication, water and oil supply, etc. are therefore optimized by intermediate storage facilities. Such type of flow is called non-conservative flow. Additionally, cost minimization- the focus of our research is integrated to such non-conservative maximum flow values pulled-out from the source. However, we work on a network with uncertain arc capabilities, as shown in Figure 2.

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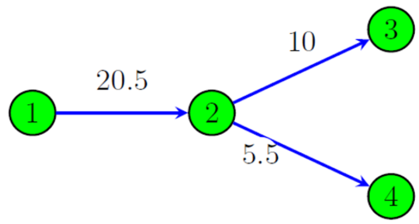

Figure 1: A Simple Instance of a Deterministic Network, Arcs with Flow Capacities

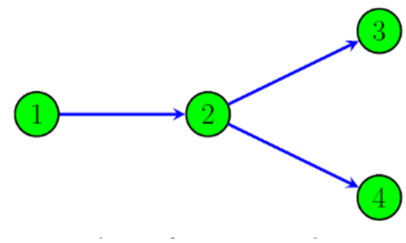

Figure 2: A Simple Instance of an Uncertain Network

The study of classical deterministic network problems has led to the development of numerous effective algorithms to maximize the flow value in various approaches, such as the fundamental simplex method proposed by Fulkerson and Dantzig [1], the augmenting path method by Ford and Fulkerson [2], and the layered network method by Dinic [3]. Using the Ford and Fulkerson algorithm, Edmonds and Karp [4] presented the flow along the shortest path. Karzanov [5] studied the preflow-push algorithm, and Goldberg and Tarjan [6] enhanced the preflow-push technique's performance. In a two terminal network, Wilkinson's [7] solution to the earliest arrival flow problem (EAFP) maximizes the flow at each time step. Minieka [8] used a pseudo-polynomial running time algorithm to solve the EAFP and a polynomial running time approach to solve the lexicographic maximum(lex-max) static flow problem. Using a genetic algorithm with flow matrices to represent each solution, Munakata and Hashier [9] addressed the MFP. The lex-max dynamic flow problem was resolved in polynomial time by Hoppe and Tardos [10]. The open shortest path first (OSPF) weight setting problem is an extension of MFP, and Ericsson et al. [11] suggested a genetic approach to solve it. Later, Gen et al. [12] suggested a genetic algorithm based on priorities to solve the MFP. Dhamala et al. [13], and Pangeni and Dhamala [14] reviewed on network optimization. Pangeni and Dhamala [15] studied flow dynamics in continuous-time with average arc capacities. Pyakurel et al. [16] studied an abstract strategy for the deterministic network evacuation problem with storage at intermediate vertices. Pyakurel et al. [17], [18], [19], [20] have also studied the network flow problems using the contraflow approach using partial, efficient, and continuous deterministic models.

In a directed network with fuzzy capacity, the maximum allowable flow was first introduced by Kim and Roush [22] who also offered some theoretical results relating to the topic. Further studies of MFP for fuzzy network can be found in [24].

The shortest path problem (SPP), one of the fundamental and crucial issues in network optimization, seeks to identify the shortest path (in terms of either time, cost or distance) between a given source vertex and a given sink vertex. The minimal cost flow problem has a number of effective algorithms. The basic version of successive shortest path method, which retains optimal solution at every step and attempts to achieve a viable flow, was independently created by Busacker and Gowen [25]. To study the MCP, primal-dual algorithm was proposed by Ford and Fulkerson [26] and capacity scaling technique

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by Edmonds and Karp [4]. To increase the effectiveness of the ϵ-relaxation method, Bertsekas and Castanon [27] proposed an auction procedure. In order to answer the minimal cost flow problem, Cai et al. [28] took into account the capacity, delay and cost of three time-varying arc weight functions. Ciurea and Ciupal [29], who modified preflow algorithms for maximum flow, introduced sequential and parallel algorithms for minimum flows. The minimal cost multicommodity flow problem in dynamic networks with time-varying capacity and arc transmission time functions were studied by Fonoberova and Lozovanu [30].

A generalization of the maximum flow of minimum cost problem for the situation of minimizing trip expenses and time were made by Mircea and Ciurea [31]. The exterior simplex type algorithm was proposed by Paparrizos et al. [32] for the cost minimization of flow. When the time horizons of the weight functions were discrete, Pyakurel [33] provided a modified minimal cost flow algorithm that computed the maximum dynamic flow and the earliest arrival flow in strongly polynomial time.

Various SPP contributions are also seen in various uncertain paradigms. An SPP in an uncertain environment views the network's associated parameters as typically nondeterministic, which can be caused by several sorts of uncertainty, such as a lack of evidence, weather, road conditions, traffic congestion and multiple sources of information from various experts etc.

Randomness was thought to be a nondeterministic phenomenon by certain scholars. They employed random variables to describe the non-deterministic properties of the issue parameters and applied probability theory to network optimization problems as a result. Frank and Hakimi [20] were the first to introduce a random network. Since then, other scholars have made major contributions to the study of random SPP, see [34], [35], [36].

The estimated probability distribution, however, is inappropriate to identify non-deterministic occurrences when the observational data are insufficient [37], [38]. The most practical and cost effective way to estimate data is to take into account the opinions of experts in order to get around this issue. The fuzzy set theory [39] is seen as one method to deal with imprecision in this context.

In order to address uncertain phenomena in human life, Liu [38] established and improved the theory of uncertainty. Liu [41] used uncertain network theory to model the scheduling issue. The maximum flow problem was initially examined by Han et al. [42] using an uncertainty theory perspective. To ascertain the expected maximum flow of an uncertain network, the authors in this case used the 99-method [41]. Gao [43] suggested two distinct models of SPP:

(i) α-shortest path and
(ii) Most shortest path.

He used Dijkstra's method to solve the crisp equivalents of these two models.

The minimal cost flow problem on an uncertain network was answered by Ding [44] by creating an algorithm.

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Ding [46] suggested α-maximum flow model of an uncertain network based on the chance-constrained model and solved the appropriate deterministic transformation of the model with a preflow-push algorithm at various confidence levels of α. Sheng and Gao [47] addressed the SPP while taking into account a two-fold uncertain hybrid environment. Shi et al. [48] also created an expected value model and a chance-constrained model of uncertain random MFP. To solve the relevant deterministic equivalents of the presented models, they used the Ford-Fulkerson algorithm. Nowadays, uncertainty theory is frequently used across many areas when data are not accessible.

A significant contribution carried out on intermediate storage flow models can be found in Dhamala and Nagurney [53]. For a review of models and algorithms for discrete evacuation planning network problems, see Dhamala [49]. Various network flow problems with excess flow storage at the intermediate vertices are addressed in [50], [51], [52].

Research Gap. The deterministic network is taken into account in the literature while discussing the MFP and SPP. In random, fuzzy or uncertain network, the issue of nonconservative flow i.e., the network flow having intermediate storage is unaddressed. Pyakurel and Dempe [54], in particular, used the network having deterministic arc capacities with storage at intermediate vertices to solve the MFP. Ding [46] suggested the α- maximum flow model of an uncertain network and Shi et al. [48] suggested the models of MFP for uncertain random network, both in conservative formulations.

So, there is a research gap in uncertain networks not having the features of non-conservative flow and in deterministic networks not addressing the uncertain behavior of the arc capacities. The SPP without accounting intermediate storage at the vertices, Ding [44] analyzed the minimal cost flow problem having uncertain arc capacities. To fill the research gap, we will focus on two issues in this study: the flow maximization problem and the cost minimization problem pertaining to the intermediate storage in the situation of the network with uncertain arc capacities.

Our Contribution. By using the lexicographic maximal flow algorithm [8] for static network, we calculate the maximum flow with intermediate storage on the network with uncertain arc capacities, which is the extension of the work of Pyakurel and Dempe [54] from the deterministic network flow problem to uncertain network flow problem. Using the maximum flow determined in the first phase of our research as the input, the second phase of our work extends the cost minimization issue of Ding [44] by incorporating intermediate storage at the network vertices with uncertain arc capacities.

For the solution of cost minimization problem, we search the shortest path in terms of cost of the flow in the arcs. While searching the minimum cost path, if the total arc capacities of the preceding arcs of an intermediate vertex exceeds the capacity of its succeeding arcs, the flow is stored at that vertex. Network simplex method is used to get the total minimum cost of the flow to the sink. The maximum flow that was drawn from the source with intermediate storage in the previous phase would therefore be included in the cost minimization. For the first time this work has been done on a network with uncertain arc capabilities with intermediate storage.

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Organization of the Paper. The structure of the article is as follows. The theory of uncertainty is discussed in Section 2. The flow models with excess flow storage that maximizes flow with minimal cost are proposed, and supporting theorem and lemma are also mentioned in this section. The development of the solution algorithms is covered in Section 3. Using a numerical example, the proposed algorithm's effectiveness is shown in Section 4. The paper is wrapped with conclusion in Section 5.

2. PRELIMINARIES

2.1 Notions on Uncertainty

In this subsection, we provide some fundamental definitions and an applicable theorem on uncertainty that are utilized in our research.

An σ-algebra is a nonempty set of subsets of the real line R that is closed under countable unions and complements. The Borel σ-algebra is the smallest σ -algebra on R that contains all open sets (or, equivalently, closed sets). Measurable sets, often known as Borel sets, are the elements of the Borel algebra.

The following notions are according to [38], [40], [41]:

Let Γ be a nonempty set, and L a σ-algebra over Γ. Each element $\wedge \in \mathrm{L}$ is called an event. A set function M from L to [0,1] is called an *uncertain measure*, if it satisfies the following axioms:

- Axiom 1. *Normality*: $M\{\Gamma\} = 1$ for the universal set Γ
- Axiom 2. *Duality*: $M\{\wedge\}$ + $\mathrm{M}\{\wedge^c\} = 1$, for any event $\wedge$
- Axiom 3. *Sub-additivity*: For every countable sequence of events $\wedge_1, \wedge_2, \ldots$ we have

$$M\ \{\cup_{i=1}^{\infty} \wedge_i\} \leq \sum_{i=1}^{\infty} M\{\wedge_i\}$$

- Axiom 4. *Product*: Let$(\Gamma_k, \mathrm{L}_k, M_k)$ be uncertainty spaces for $k = 1, 2, \ldots, n$. Then, the product uncertain measure M is an uncertain measure on the product σ-algebra $L_1 \times L_2 \times \ldots \times L_n$ satisfying

$$M\{\prod_{k=1}^{n} \wedge_\mathrm{k}\} = \min_{1 \leq \mathrm{k} \leq \mathrm{n}} \mathrm{M_k}\{\wedge_\mathrm{k}\}$$

An *uncertain variable* is a measurable function ξ from an uncertainty space (Γ, L, M) to the set of real numbers such that $\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\}$ is an event for any Borel set B of real numbers.

The uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ are said to be *independent* if,

$$M\{\bigcap_{i=1}^{n} (\xi_i \in B_i)\} = \bigwedge_{i=1}^{n} M\{\xi_i \in B_i\}$$

For any Borel sets $B_1, B_2, \ldots, B_n$ of real numbers.

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The *uncertainty distribution* Φ of an uncertain variable ξ is defined by

$$\Phi(x) = M\{\xi \le x\};\ \forall x \in R.$$

An uncertain variable ξ is called *zigzag*, if it has a zigzag uncertainty distribution

$$\phi(x) = \begin{cases} 0, & x \le a \\ \dfrac{x-a}{2(b-a)}, & a \le x \le b \\ \dfrac{x+c-2b}{2(c-b)}, & b \le x \le c \\ 1, & x \ge c \end{cases}$$

And denoted by Z(a, b, c), where a, b, c are real numbers with a < b < c.

The inverse uncertainty distribution of zigzag uncertain variable Z(a, b, c) is

$$\phi^{-1}(\beta)=\begin{cases}(1-2\beta)a+2\beta b, & \beta < 0.5\\ (2-2\beta)b+(2\beta-1)c, & \beta \ge 0.5\end{cases}$$

An uncertainty distribution $\Phi(x)$ is said to be *regular*, if it is a continuous and strictly increasing function with respect to x at which $0 < \phi(x) < 1$, and $\lim_{x\to-\infty}\phi(x) = 0,\ \lim_{x\to\infty}\phi(x) = 1.$

Theorem 1. *Let $\xi_1, \xi_2, \dots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\phi_1, \phi_2, \dots, \phi_n$ respectively. If the function $(x_1, x_2, \dots, x_n)$ is strictly increasing with respect to $x_1, x_2, \dots, x_n$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \dots, x_n$ then the uncertain variable, $\xi = g(\xi_1, \xi_2, \dots, \xi_n)$ has an inverse uncertainty distribution,*

$$\varphi^{-1}(\beta)= g(\phi_1^{-1}(\beta),\ \phi_2^{-1}(\beta), \dots, \phi_m^{-1}(\beta), \phi_{m+1}^{-1}(1-\beta), \phi_{m+2}^{-1}(1-\beta), \dots, \phi_n^{-1}(1-\beta)).$$

2.2 Mathematical Formulation

Consider a directed network N = (V, A) that has a set of vertices V and a set of arcs A with uncertain arc capacities, where the vertices are the points where the arcs cross each other i.e., $A \subset V \times V$ with |V| = n. Let the flow value in the arc (i, j) be $f_{ij} \ge 0$. Also, the flow in the arc (i, j) has an upper bound u_{ij} , and the cost of a unit flow is w_{ij}. Let s and t represent, respectively, the flow starting vertex (source) and the flow terminating vertex (sink).

Let storage $s_i \ge 0$ be allocated at vertex i. additionally, the storage capacity and supply/demand at the vertex i are v_i and m_i, respectively. Here, m_i is taken to be positive for the flow supplying vertices and negative for the flow demanding vertices. Assume that F, which is a function of ξ_{ij} , the uncertain arc capacity of an arc (i, j), is the entire flow from the source (beginning) vertex s to the sink (terminating) vertex t in the uncertain network Ñ. The uncertainty distribution and it's inverse corresponding to the uncertain variable ξ_{ij} are ϕ_{ij} and ϕ_{ij}^{-1}, respectively. The notations used in this paper are listed in Table 1 for convenient reference.

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Table 1: The Common Notations

N	A deterministic network	**R**	The set of real numbers
Ñ	An uncertain network	L	σ-Algebra over Γ, where Γ is a nonempty set
V	The set of vertices	Λ	An event on L
A	The set of arcs	M	An uncertain measure on L
$\mathbf{F_{ij}}$	The amount of flow in arc (i, j)	(Γ, L, M)	An uncertainty space
u_{ij}	The capacity of flow in arc (i, j)	ξ	An uncertain variable
$\mathbf{w_{ij}}$	The cost of unit flow in arc (i, j)	B	A Borel set
s, t	The source and sink, respectively	ϕ	The uncertainty distribution
$\mathbf{s_i}$	The amount of flow stored at vertex i	ϕ^{-1}	The inverse uncertainty distribution
$\mathbf{v_i}$	The capacity of flow storage at vertex i	β	The confidence level
$\mathbf{m_i}$	The supply/demand of vertex i	F	The total flow from the source

We now divide our problem into two parts. The flow maximization problem with deterministic intermediate storage and uncertain arc capacity is formulated in Phase I. After that, in Phase II, we apply the flow of Phase I to the cost minimization issue in uncertain network.

Phase I: Flow maximization model. *Max-Flow*: (with excess flow storage)

$$\max\left[F + \sum_{i\in V\backslash\{s,t\}} s_i\right] \quad (1)$$

$$\sum_{j:(s,j)\in A} f_{sj} - \sum_{j:(j,s)\in A} f_{js} = F \quad (2)$$

$$\sum_{j:(i,j)\in A} f_{ij} - \sum_{j:(j,i)\in A} f_{ji} + s_i = 0 \ \forall\ i \in V\backslash\{s,t\} \quad (3)$$

$$\sum_{j:(t,j)\in A} f_{tj} - \sum_{j:(j,t)\in A} f_{jt} = -\left(F - \sum_{j\in V\backslash\{s,t\}} s_j\right) \quad (4)$$

$$0 \le f_{ij} \le \xi_{ij}, (i,j) \in A \quad (5)$$

$$0 \le s_i \le v_i, i \in V \quad (6)$$

Maximizing the total flow out from the starting vertex (source) is the objective function (1)'s main goal. The total mass supplied by the source s is given by equation (2). The mass balance for the intermediate vertices is given by equation (3). The total mass demanded by the sink t is given by equation (4). The capacity constraints of the arcs and vertices, respectively, are given by inequations (5) and (6).

The constraints in (5) can be rewritten as the constraints in (13), and using the Theorem 2, the rewritten constraints may be transformed into deterministic capacity constraints in (7). The remodeling of *Max-Flow* model maximizes the objective function (1) with respect to the constraints (2)-(4), (6) and the following constraints (7).

$$0 \le f_{ij} \le \phi_{ij}^{-1}(1-\beta), \quad (i,j) \in A \quad (7)$$

Phase II: Cost minimization model. To solve the SPP in terms of cost, the excess flow must be stored at the intermediate vertices, when the total flow entering a vertex exceeds the total capacity of the arcs originating from that vertex. In this case, the deterministic cost minimization (*Min-Cost-D*) model with excess flow storage is formulated as follows:

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Min-Cost-D: (with excess flow storage)

$\min \sum_{(i,j)\in A} w_{ij}\ f_{ij}$ (8)

$\sum_{j:(i,j)\in A} f_{ij} = m_i\,, i = s, m_i\ > 0$ (9)

$\sum_{j:(i,j)\in A} f_{ij} - \sum_{j:(j,i)\in A} f_{ji} + s_i = 0\ \ \forall\ i \in V\backslash\{s,t\}$ (10)

$0\ \le f_{ij} \le u_{ij}\,, (i,j) \in A$ (11)

$0\ \le s_i \le v_i\,, i \in V$ (12)

The goal of the objective function (8) is to reduce overall flow costs. The mass supplied from the source is provided by constraints (9). However, in other circumstances the same amount of demand for the sink would not be satisfied due to the network's minimum cut capacity. Equations (10) refer to intermediate storage restrictions. Constraints (11) and (12) provide limits on the capacity of arcs and vertices, respectively.

Traditional minimum cost flow problems presuppose that the arc's capacity is fixed. However, in reality, it is not. Because the probability distribution of arc capacity cannot be constructed due to uncertainty factors or a lack of data, the believe degree approach is successfully applied even though there is a lack of data.

The *Min-Cost-D* model can be reformulated to the uncertain network Ñ = (V, A, ξ) with uncertain arc capacity ξ = {ξ_{ij} / (i, j) ∈ A} and uncertain measure constraints with a certain confidence level β. This *reformulated model* minimizes the objective function (8) with respect to the constraints (9), (10), (12) and the following constraints (13).

$M\{f_{ij} \le \xi_{ij}\}\ \ge \beta, \quad (i,j) \in A$ (13)

The constraints (13) mean that the flows need to satisfy flow bound with a given confidence level β.

The following theorem is the conversion theorem for the constraints (13).

Theorem 2. *In a uncertain network* Ñ = *(V, A, ξ), let* ξ_{ij} *be independent uncertain variables with regular uncertain distribution* $\phi(x)$*, for (i, j) ∈ A. Then, the constraints (13) are equivalent to the inequalities*

$f_{ij} \le \phi_{ij}^{-1}(1-\beta)\,, \quad (i,j) \in A$ (14)

Proof. Since, the measure of uncertain variable is self-dual, we have

$$M\{f_{ij} \le \xi_{ij}\} +\ M\{f_{ij} \ge \xi_{ij}\} = 1\ , \quad (i,j) \in A$$

Using the constraints (13) in this relation, we have

$$M\{f_{ij} \le \xi_{ij}\} = 1 -\ M\{f_{ij} \ge \xi_{ij}\} \ge \beta\ , \quad (i,j) \in A$$

This implies

$$M\{f_{ij} \ge \xi_{ij}\} \le 1 -\ \beta\ \Rightarrow f_{ij} \le \phi_{ij}^{-1}(1-\beta), \quad (i,j) \in A$$

Hence, the theorem is established.

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Therefore, the *newly reformulated model* of *Min-Cost-D* minimizes the objective function (8) with respect to the constraints (9), (10), (12) and the constraints (14) using many efficient solution methods. Furthermore, the uncertainty distribution of the minimum cost can be created by selecting various values of β.

The following theorem and lemma further illustrate the properties of the latest deterministic model.

Theorem 3. [44] Let ξ_{ij} be independent uncertain variables with regular uncertainty distributions $\phi(x)$, for all arc (i, j) ∈ A, respectively. Then, the newly reformulated model subject to constraint (13) is non-decreasing with respect to the confidence level β.

Lemma 1. [44] If the feasible set of the newly reformulated model provided by constraint (13) is empty for β_0, then it is empty for any $\beta > \beta_0$.

This lemma gives permission to choose the bisection method to find the greatest confidence level β for a feasible flow.

3. SOLUTION ALGORITHM

For the flow maximization and cost minimization flow problems, both in the context of intermediate storage and uncertain environment, we provide two explicit approaches. When taking into account intermediate storage, the first algorithm performs well in obtaining the maximum flow from the source.

ALGORITHM 1 [45] Uncertain Measure Based Maximum Flow with Excess Flow Storage

- **Input:** Given an uncertain network, $\tilde{N} = (V, A, f_{ij}, \xi_{ij}, \phi_{ij})$, $0 < \beta < 1$.
  1. Evaluate $\phi_{ij}^{-1}(1-\beta), \quad (i,j) \in A$.
  2. Assign the capacities of the associated arcs in the deterministic network to the values from Step 1.
  3. Take the length of each arc as unit and find the shortest distance between the intermediate vertices from the source, where the flow to vertex i violates the capabilities of the succeeding arcs i.e., $\sum_{(j,i)\in A} f_{ji} > \sum_{(i,j)\in A} \phi_{ij}^{-1}(1-\beta)$.
  4. Prioritize the vertices of Step 3 more highly which are farthest from the source.
  5. Along with the provided sink, turn the vertices from Step 3 into virtual ones and use those as sinks.
  6. From the priority of Step 4, determine the maximum flow in the modified network with the single source and multiple sinks.
  7. To acquire the maximum flow at the specified sink and storage at the specified intermediate vertices, turn on the network N solution while turning off the virtual vertices and arcs.
- **Output:** Flow with the maximum value in the uncertain network Ñ.

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Theorem 4. *The maximum flow with intermediate storage at the intermediate vertices obtained by the above Algorithm 1 is feasible as well as optimal and has polynomial time complexity.*

Proof. The following three steps are how we prove the theorem.

Feasibility. By employing the associated inverse distributions of the uncertain variables with various values of β, the values of Step 1 may be evaluated. Step 2 is possible since it is an extract of Theorem 1 and uses the reformulated model of *Max-Flow* model. The distance of each intermediate vertex and of the sink can be calculated by counting the number of vertices (one arc is equivalent to one unit distance) from the source to other vertices, and the priority order can be assigned to the intermediate vertices where capacity violation of the successor arcs occur. Steps 3 and 4 are therefore feasible. Step 5 is possible since the creation of virtual arcs and vertices do not go against capacity restrictions. It is also possible to figure out the maximum flow using Step 4's priority order. Turning off the virtual vertices and arcs take feasibility into account.

Optimality. The maximum flow value is estimated to the sink with the highest priority after creating virtual vertices with virtual arcs and applying a flow conservation restriction at the intermediate vertices. The flow is directed toward the sink in the residual network up until the appearance of the augmenting path. Keep in mind that the length of the backward arcs is regarded to be negative compared to the length of the equivalent forward arcs. The previous method is used to compute the greatest flow to the intermediate vertex in the following iteration, which is designated as the sink in this iteration and has the second-largest distance, or the second largest arc length from the source. Following this approach, we obtain the optimal flow in the network with virtual vertices and arcs without the need for intermediate storage, according to Minieka [8], and Pyakurel and Dempe [54]. The reformulated model of Max-Flow model achieves the required result by shutting off the virtual vertices and arcs after returning the flow to their corresponding vertices.

Complexity. The proposed Algorithm 1 has polynomial time complexity, just like the algorithm from Minieka [8].

As a result, the theorem is established.

The next algorithm, which is carried out after using the previous algorithm, works well to achieve the minimum flow cost to the maximum flow extracted from the source in non-conservative flow case.

ALGORITHM 2: Minimum Cost with Uncertain Arc Capacities with Excess Flow Storage

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- **Input:** Given Ñ = (V, A, f_{ij}, ξ_{ij}, w_{ij}, φ_{ij}, s_i, v_i, m_i), β ∈ (0, 1) and an error tolerance, ϵ.
  1. Use the bisection method to set β, if 1 − 0 > ϵ. Go to Step 2 after creating a deterministic network with each arc's capacity
  2. set to $u_{ij} = \phi_{ij}^{-1}(1-\beta)$. If not, proceed to Step 3.
  3. Obtain the first feasible solution (use Big M method). If a feasible solution cannot be found, set β = 1 and go to Step 1. Otherwise, proceed to Step 1 and set β = 0.

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4. Set β = 0, make the deterministic network N = (V, A). Also, set $u_{ij} = \phi_{ij}^{-1}(1-\beta)$.
5. Prepare the priority ordering of the sink's access routes based on their lowest cost.
6. According to the order of Step 4, direct the flow to the direction of the sink.
7. When $\sum_{(j,i)\in A} f_{ji} > \sum_{(i,j)\in A} u_{ij}$, store the surplus flow at vertex i.
8. Use the network simplex technique to get the minimal total flow cost $\sum_{(i,j)\in A} w_{ij}\ f_{ij}$ and the β-minimum cost flow in the network N.

- **Output:** Uncertain minimum cost flow with predetermined confidence level β.

Theorem 5. *Algorithm 2 computes the minimum cost with polynomial time complexity to the maximal flow computed by the Algorithm 1 in uncertain network.*

Proof. The following are the steps how we prove the theorem.

Feasibility and Optimality. Lemma 1 permits to use the bisection method within the domains of confidence level β and the error tolerance ϵ. Theorem 2 converts the uncertain network to the network with deterministic arc capacities so that the optimum flow can be sent toward the sink respecting the arcs capacities. Theorem 3 describes non decreasing feature of *newly reformulated model* with constraints (13) with respect to confidence level β. Network simplex algorithm, one of the fastest algorithm can solve the *newly reformulated model*. The priority ordering of the sink's access routes is based on their lowest cost. To overcome the minimum cut scenario of the network, virtual nodes and arcs with zero costs are created to store the excess flow. Later, the stored flow is sent back to the corresponding intermediate

vertices which gives solution to the *newly reformulated model*.

Complexity. The proposed Algorithm 2 has polynomial time complexity, just like the algorithm of Minieka [8].

Hence, the theorem is proved.

4. ILLUSTRATIVE EXAMPLE

Maximum Flow with Intermediate Storage. As seen in Figure 3, an uncertain network Ñ = (V, A, ξ) is considered. Here, the source and sink are taken as the vertices 1 and 4, respectively. The flow capacity $\phi_{ij}^{-1}(1-\beta)$ for randomly taken value of confidence level β = 0.7, β ∈ (0, 1) are calculated and unit flow costs w_{ij} of the arcs are provided in Table 2. $\phi_{ij}^{-1}(1-\beta)$ is assumed constant for constant value of ξ_{ij}.

The maximum flow value with intermediate storage for β = 0.7 is 7.2 and comes from the source [see Figure 4]. For different values of confidence level β, the flow paths of maximum flow values and the corresponding intermediate storage values are presented in Table 3. The uncertainty distribution of the maximum flow with and without intermediate storage is presented in Figure 6. This figure depicts the benefit of non-conservative flow

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over the conservative flow. Also, we can choose appropriate values of the confidence level for desired value of the optimal flow.

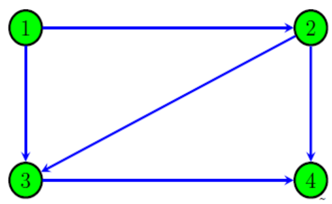

Figure 3: A Simple Instance of Uncertain Network Ñ = (V, A, ξ)

Table 2: The Per Unit Flow Cost and Capacity Information of Uncertain Network

Arc	ξ_{ij}	w_{ij}	$\Phi_{ij}^{-1}(1-\beta)$ $\beta \le 0.5$	$\beta > 0.5$	Value of $\Phi_{ij}^{-1}(1-0.7)$
(1, 2)	$\mathcal{Z}(4, 5, 6)$	10	$6-2\beta$	$6-2\beta$	4.6
(1, 3)	$\mathcal{Z}(2, 3, 5)$	2	$5-4\beta$	$4-2\beta$	2.6
(2, 3)	$\mathcal{Z}(2,3,4)$	1	$4-2\beta$	$4-2\beta$	2.6
(2, 4)	4.4	6	4.4	4.4	4.4
(3, 4)	$\mathcal{Z}(1,2,3)$	3	$3-2\beta$	$3-2\beta$	1.6

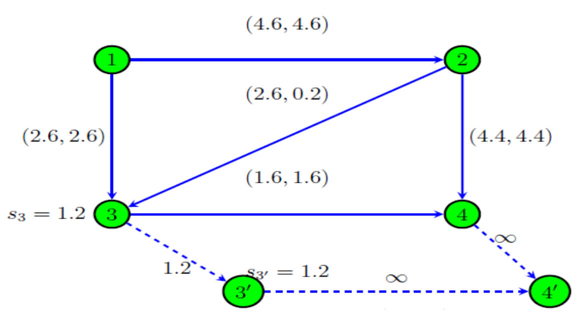

Figure 4: β= 0.7 Maximum Flow, Arc with Capacity and Flow Value (u_{ij}, f_{ij}), Vertex i with Storage s_i and Dashed arc are Virtual with Flow Capacity (Intermediate Storage Case). The Maximum Flow F = 7.2.

Minimum Cost with Intermediate Storage. The total minimum cost flow to the vertex 4 (which is taken as sink) for randomly taken β = 0.7 with storage at the intermediate vertices is represented in the Figure 5 and for other values of β is calculated in Table 3. Since the objective function is linear, the total minimum flow cost is calculated using the network simplex algorithm or any other suitable method can also be used. From among the minimum costs of various paths, the smallest (in value) cost and its corresponding path can be chosen as the required objective.

The uncertainty distribution of total minimum cost is shown in Figure 7. This figure shows the inverse variation of the total minimal cost with the values of confidence level.

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Additionally, we can determine the entire minimum flow costs to every vertex other than the sink for various confidence level β values and obtain the desired minimal cost.

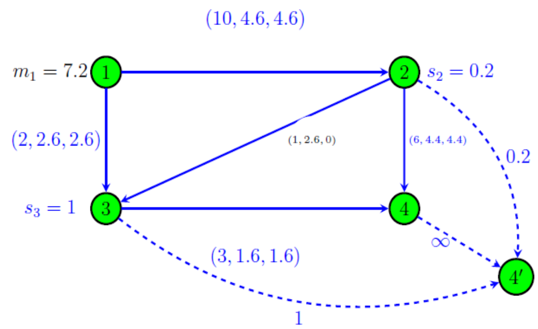

Figure 5: β= 0.7 Minimum Cost Flow Path, arc with Unit Flow Cost, Capacity and Flow Value (w_{ij}, u_{ij}, f_{ij}) (Excess Flow Storage Case), Dashed Arcs are Virtual with Flow Capacity, $\sum w_{ij}f_{ij}$ = 82.4, and Maximum F = 6+1.2 = 7.2

Table 3: The Flow Path, Flow Value and Minimum Cost in uncertain Network for Different Confidence Levels

	Flow path	Flow value	Storage	Flow cost $w_{ij}f_{ij}$
$\beta = 0.1$				
	$1 \to 2 \to 4$	$f_{12} = 5.8,\ f_{24} = 4.4$		$58 + 26.4 = 84.4$
	$1 \to 3 \to 4$	$f_{13} = 4.6,\ f_{34} = 2.8$	$s_3 = 3.2$	$9.2 + 8.4 = 17.6$
	$1 \to 2 \to 3 \to 4$	$f_{23} = 1.4$		1.4
	values $\Rightarrow$	$\sum f_{1j} = 10.4, \sum f_{j4} = 7.2$	$\sum s_i = 3.2$	$\sum w_{ij}f_{ij} = 103.4$
$\beta = 0.3$				
	$1 \to 2 \to 4$	$f_{12} = 5.4,\ f_{24} = 4.4$		$54 + 26.4 = 80.4$
	$1 \to 3 \to 4$	$f_{13} = 3.8,\ f_{34} = 2.4$	$s_3 = 2.4$	$7.6 + 7.2 = 14.8$
	$1 \to 2 \to 3 \to 4$	$f_{23} = 1$		1
	values $\Rightarrow$	$\sum f_{1j} = 9.2,\ \sum f_{j4} = 6.8$	$\sum s_i = 2.4$	$\sum w_{ij}f_{ij} = 96.2$
$\beta = 0.5$				
	$1 \to 2 \to 4$	$f_{12} = 5,\ f_{24} = 4.4$		$50 + 26.4 = 76.4$
	$1 \to 3 \to 4$	$f_{13} = 3,\ f_{34} = 2$	$s_3 = 1.6$	$6 + 6 = 12$
	$1 \to 2 \to 3 \to 4$	$f_{23} = 0.6$		0.6
	values $\Rightarrow$	$\sum f_{1j} = 8,\ \sum f_{j4} = 6.4$	$\sum s_i = 1.6$	$\sum w_{ij}f_{ij} = 89$
$\beta = 0.7$				
	$1 \to 2 \to 4$	$f_{12} = 4.6,\ f_{24} = 4.4$	$s_2 = 0.2$	$46 + 26.4 = 72.4$
	$1 \to 3 \to 4$	$f_{13} = 2.6,\ f_{34} = 1.6$	$s_3 = 1$	$5.2 + 4.8 = 10$
	values $\Rightarrow$	$\sum f_{1j} = 7.2,\ \sum f_{j4} = 6$	$\sum s_i = 1.2$	$\sum w_{ij}f_{ij} = 82.4$
$\beta = 0.9$				
	$1 \to 2 \to 4$	$f_{12} = 4.2,\ f_{24} = 4.2$		42 +25.2 =67.2
	$1 \to 3 \to 4$	$f_{13} = 2.2,\ f_{34} = 1.2$	$s_3 = 1$	4.4 +3.6 =8
	values $\Rightarrow$	$\sum f_{1j} = 6.4,\ \sum f_{j4} = 5.4$	$\sum s_i = 1$	$\sum w_{ij}f_{ij} = 75.2$

5. CONCLUSION

For a variety of reasons, the network parameters in network optimization issues might not be deterministic. Different theories, such as fuzzy theory, uncertainty theory, and probability theory, assist us in overcoming these challenges. Probability theory provides answers to network optimization problems when the parameters exhibit random characteristics. Probability distributions cannot be created and probability theory is invalid

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due to insufficient data. The development of fuzzy theory and uncertainty theory is necessary to address issues in such non-random environments. These theories employ the appropriate measure to address the relevant issues. The measure of union of events is where the conflict between the possibility measure of fuzzy theory and the uncertain measure of uncertainty theory resides. They are utilized in many ways. Here, we applied the uncertainty theory's uncertain measure to network flow problem.

The traditional network flow problems of flow maximization and cost minimization on deterministic arc capacities were expanded in this research to the same problems with non-conservative flow features having uncertain environments in arcs capacities. The main difference between conservative and non-conservative flow problems is the concern of storage facility of the flow at the intermediate vertices. In intermediate storage scenarios, we first developed the flow maximization model, then the cost minimization flow model, both in uncertain arc capacity environment and are then converted to their deterministic equivalents by utilizing the uncertainty theory. The maximized flow is pulled out in lexicographic order from the source to the sink by developing an efficient Algorithm 1 with a similar level of complexity to that of earlier literary works. This maximized flow was used as the input and Algorithm 2 is developed to send the flow to the sink via the path having minimum cost. So, we optimized the flow value in the first phase and then the cost in the second phase. Domination of the non-conservative flow value over the conservative flow value are depicted graphically. 19% to 44% flow values for different values of confidence level are increased in non-conservative flow, which verifies the benefit of storage facility at the intermediate vertices, see Figure 6. The inverse variation of the minimum cost and both type (conservative and non-conservative) of maximum flow values with the confidence level values are also depicted in Figures 6 and 7. This variation can be used to get the desired value of the flow and the flow cost for a specific value of confidence level. For example; for the confidence level 0.5, the non-conservative flow value is 8 and the minimum cost is 89. Such type of problems, which have uncertainty in real life, are relevant both conceptually and practically.

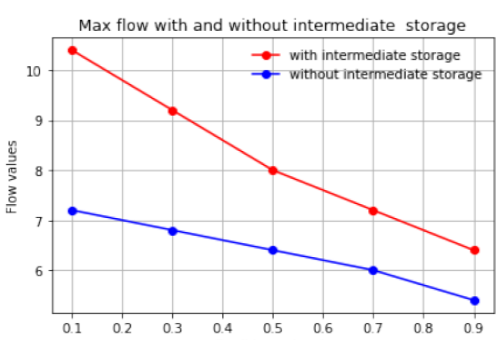

Figure 6: Uncertainty Distribution of the Max Flow with and without Intermediate Storage

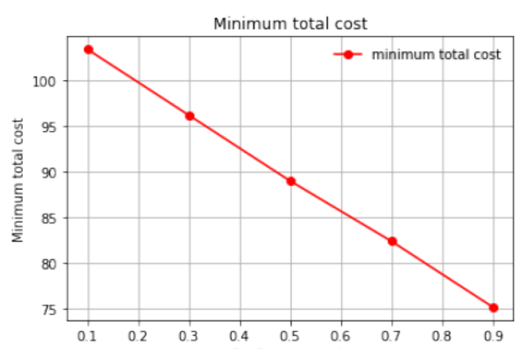

Figure 7: Uncertainty Distribution of the Minimum Total Cost

Data Availability. The study was not supported by any additional data.

Conflicts of Interest. The authors declare that there is no conflict of interest regarding the publication of this paper.

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