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NUMERICAL SOLUTION OF INTEGRO-DIFFERENTIAL EQUATIONS OF THE SECOND KIND BY USING GALERKIN'S METHOD WITH GENOCCHI POLYNOMIAL

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Abstract

The purpose of this work is to search for an approximate solution to the Fredholm and Volterra integro-differential equations using Genocchi polynomials, replacing the initial conditions if necessary, where the integrals can be calculated using numerical methods, in order to obtain a variation problem and reduce it to a linear system, where its solution is to find the coefficients of the function. Unknowns and then solve the equation. The convergence and effectiveness of this method are confirmed by numerical examples that will be presented.

Keywords: Integro-differential Equations; Galerkin Method; Genocchi Polynomials.

1. INTRODUCTION

Integro-differential equations are considered one of the most important fields in mathematical disciplines, for example pure mathematics and applied mathematics. Integro-differential equations have a very important role in modern science and technology such as heat transfer, diffusion processes, mechanics, biological species, and many other fields. To learn more about the sources in which these types of equations are studied in applications of physics, biology, and engineering, as well as in books on advanced integral equations. References can be found [4, 10, 11, 15].

The numerical solution of second order integro-differential equations with the boundary conditions of the Fredholm and Volterra equations and other equations related to this type of equations has been done by some authors.

For example, the authors in [4] discussed the Chebyshev Collocation Method for the Solution of Linear Integro-Differential Equations, which is the compact finite difference method, and the monotonic iterative sequence method for solving the second order Volterra integro-differential equation was implemented in [27,5]. However, a sequential solution of second order integro-differential equations with boundary conditions of Fredholm and Volterra types by the homotopy analysis method was also considered in [16].

Accordingly, this work aims to find approximate solutions to Fredholm and Volterra linear integro-differential equations of the second type using polynomials of the Genocchi type with the Galerkin method and then compare the approximate solutions with the exact solutions to see the effectiveness of the method through the examples that we will present.

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2. GENOCCHI GALERKIN METHOD FOR IDE

Consider the following Volterra integro-differential equation

$$u'(x) = f(x) + \int_a^x k(x,t)u(t)\,dt \quad (1)$$

$$u(a) = \propto \quad (2)$$

Where $f(x)$ and $k(x,t)$ are known functions, and $u(x)$ is the unknown function that must be determined**.**

The method under study uses Genocchi polynomials, well addressed in, [12,22,14,13].

As a basis polynomial to approximate the solution on a closed finite interval. Assume that

$$u(x) = u_n(x) = \sum_{i=0}^{n} \beta_i\, G_i(x) = \sum_{i=0}^{n} \beta_i\, G_i\left(\frac{x-a}{b-a}\right) \quad (3)$$

Where $G_i\left(\frac{x-a}{b-a}\right)$ is shifted Genocchi polynomial at $[a,b]$

Note that when we take the value $x = a$ we get $\frac{x-a}{b-a} = 0$, and when $x = b$ we get $\frac{x-a}{b-a} = 1$.

So we have

$$u'(x) = u_n'(x) = \sum_{i=0}^{n} \left(\frac{1}{b-a}\right) \beta_i G_i'\left(\frac{x-a}{b-a}\right) \quad (4)$$

Substituting (3) and (4) into (1), results in

$$\sum_{i=0}^{n} \left(\frac{1}{b-a}\right) \beta_i G_i'\left(\frac{x-a}{b-a}\right) = f(x) + \int_a^x k(x,t) \sum_{i=0}^{n} \beta_i G_i\left(\frac{t-a}{b-a}\right) dt = f(x) + \sum_{i=0}^{n} \beta_i \int_a^x k(x,t) G_i\left(\frac{t-a}{b-a}\right) dt \quad (5)$$

To determine unknown coefficients β_i, we use the Galerkin idea by multiplying both sides of (5) by $G_j\left(\frac{t-a}{b-a}\right)$ and then integrating with respect to x from 0 to 1. So we have

$$\sum_{i=0}^{n} \left(\frac{1}{b-a}\right) \beta_i \int_0^1 G_i'\left(\frac{x-a}{b-a}\right) G_j\left(\frac{x-a}{b-a}\right) dx = \int_0^1 f(x)\, G_j\left(\frac{x-a}{b-a}\right) dx + \int_0^1 \left[\sum_{i=0}^{n} \beta_i \int_a^x k(x,t) G_i\left(\frac{t-a}{b-a}\right) dt\right] G_j\left(\frac{x-a}{b-a}\right) dx \quad (6)$$

for $= 0,1,\dots,n$, or equivalently

$$\sum_{i=0}^{n} \left(\frac{1}{b-a}\right) \beta_i \int_0^1 G_i'\left(\frac{x-a}{b-a}\right) G_j\left(\frac{x-a}{b-a}\right) dx = \int_0^1 f(x)\, G_j\left(\frac{x-a}{b-a}\right) dx + \sum_{i=0}^{n} \beta_i \int_0^1 \left[\int_a^x k(x,t) G_i\left(\frac{t-a}{b-a}\right) dt\right] G_j\left(\frac{x-a}{b-a}\right) dx \quad \textbf{(7)}$$

If needed the integrals can be calculated by numerical methods. This procedure generates a system of linear equations for the unknown $\{\beta_0, \beta_1, \dots, \beta_n\}$. Many researchers substitute initial condition

$$u(a) = \propto \Rightarrow \sum_{i=0}^{n} \beta_i G_i\left(\frac{a-a}{b-a}\right) = \sum_{i=0}^{n} \beta_i G_i\,(0) = \propto \quad (8)$$

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for the same number of equations in the foregoing linear system. The unknown parameters are determined by solving the system of equations (7) and (8). Substituting these values in (3) gives the approximate solution of the integro-differential equation (1).

3. GENOCCHI POLYNOMIALS AND THEIR PROPERTIES

The classical Genocchi polynomial $G_n(x)$ is usually defined by means of the exponential generating functions

$$\frac{2te^{xt}}{e^t + 1} = \sum_{n=0}^{+\infty} G_n(x)\frac{t^n}{n!}$$

Where $G_n(x)$ is the Genocchi polynomial of degree n and is given by

$$G_n(x) = \sum_{k=0}^{n} \binom{n}{k} G_{n-k}(x)\, x^k$$

G_{n-k}. Is the Genocchi number.

Some of the important properties of these polynomials include

$$\begin{cases} \int_0^1 G_p(x)G_q(x)\,dx = \dfrac{2(-1)^p p!\,q!}{(p+q)!} G_{p+q}, p, q \in \mathbb{N}^* \\ \dfrac{dG_p(x)}{dx} = nG_{n-1}(x),\ \ p \in \mathbb{N}^* \\ G_p(1) + G_q(0) = 0,\ \ p \in \mathbb{N}^* \end{cases}$$

Noting that, the Genocchi polynomial $G_n(x)$ is a polynomial with rational coefficients.

$$G_0(x) = 0, G_1(x) = 1, G_2(x) = 2x - 1, G_3(x) = 3x^2 - x, G_4(x) = 4x^3 - 6x^2 + 1, \ldots$$

4. NUMERICAL EXAMPLES

In this section, we intend to show the efficiency of the Galerkin method for solving Fredholm and Volterra integro-differential equations of the second kind by Genocchi polynomials by presenting six illustrative examples. The absolute error for this formulation is de ned by $E(x) = |u(x) - u_n(x)|$.

Example 1. *Let us consider the linear integro-differential equation of Volterra*.

$$u'(x) = 2 - \frac{x^2}{4} + \frac{1}{4}\int_0^x u(t)dt$$

With the initial condition $u(0) = 0$

Where the function $f(x)$ *is chosen such that the exact solution is given by*

$$u(x) = 2x$$

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The approximate solution $u_n(x)$ of $u(x)$ is obtained by the Galerkin-Genocchi polynomial method.

Table 1: We Present the Exact and Approximate Solutions of the Equation in example 1 in Some Arbitrary Points, the error for N=10.

x	***exact solution*** u	***approximate solution*** u_n	***Error***
0.0	0.0000	0.0000	0.000000e+00
0.1	0.2000	0.2000	0.000000e+00
0.2	0.4000	0.4000	0.000000e+00
0.3	0.6000	0.6000	0.000000e+00
0.4	0.8000	0.8000	0.000000e+00
0.5	1.0000	1.0000	0.000000e+00
0.6	1.2000	1.2000	0.000000e+00
0.7	1.4000	1.4000	0.000000e+00
0.8	1.6000	1.6000	0.000000e+00
0.9	1.8000	1.8000	0.000000e+00
1	2.0000	2.0000	0.000000e+00

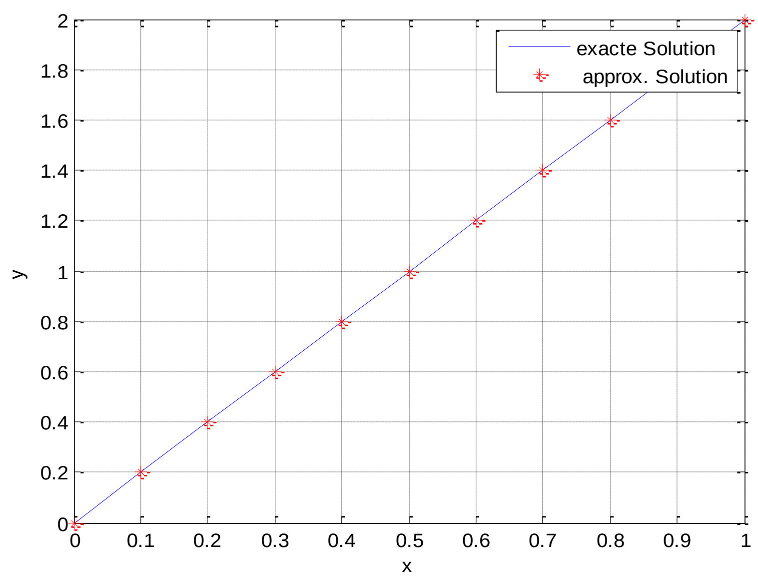

Figure 1: Graph for example 1

Example 2. *Let us consider the linear integro-differential equation of Volterra.*

$$u'(x) = 1 - 2x\sin(x) + \int_0^x u(t)\,dt$$

With the initial condition $u(0) = 0$

Where the function $f(x)$ *is chosen such that the exact solution is given by*

$$u(x) = x\cos(x)$$

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The approximate solution $u_n(x)$ of $u(x)$ is obtained by the Galerkin-Genocchi polynomial method.

Table 2: We Present the Exact and Approximate Solutions of the Equation in example 2 in Some Arbitrary Points, the Error for N=9

x	***exact solution*** u	***approximate solution*** u_n	***Error***
0.0	0.0000	0.0000	0.000000e+00
0.1	0.0995	0.0995	3.500639e-10
0.2	0.1960	0.1960	4.629055e-10
0.3	0.2866	0.2866	5.292684e-10
0.4	0.3684	0.3684	3.588525e-10
0.5	0.4388	0.4388	4.901383e-10
0.6	0.4952	0.4952	6.156364e-10
0.7	0.5354	0.5354	4.568667e-10
0.8	0.5574	0.5574	5.604863e-10
0.9	0.5594	0.5594	5.604863e-10
1	0.5403	0.5403	1.096326e-09

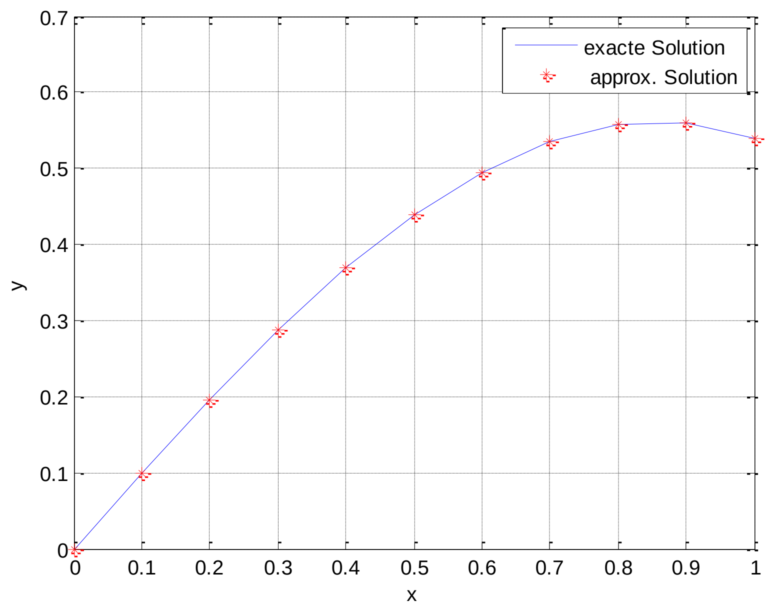

Figure 2: Graph for example 2

Example 3. *Let us consider the linear integro-differential equation of Fredholm*

$$u'(x) = 3e^{3x} - \frac{1}{3}(2e^3 + 1)x + \int_0^1 3xtu(t)\,dt$$

With the initial condition $u(0) = 1$

Where the function $f(x)$ *is chosen such that the exact solution is given by*

$$u(x) = e^{3x}$$

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The approximate solution $u_n(x)$ of $u(x)$ is obtained by the Galerkin-Genocchi polynomial method.

Table 3: We Present the Exact and Approximate Solutions of the Equation in example 3 in Some Arbitrary Points, the Error for N=9

x	***exact solution u***	***approximate solution u_n***	***Error***
0.0	1.0000	1.0000	0.000000e+00
0.1	1.3499	1.3499	4.500060e-06
0.2	1.8221	1.8221	5.619143e-06
0.3	2.4596	2.4596	7.016613e-06
0.4	3.3201	3.3201	5.099747e-06
0.5	4.4817	4.4817	6.679660e-06
0.6	6.0496	6.0496	8.956952e-06
0.7	8.1662	8.1662	7.585738e-06
0.8	11.0232	11.0232	8.937670e-06
0.9	14.8797	14.8797	1.159044e-05
1	20.0855	20.0855	1.697252e-05

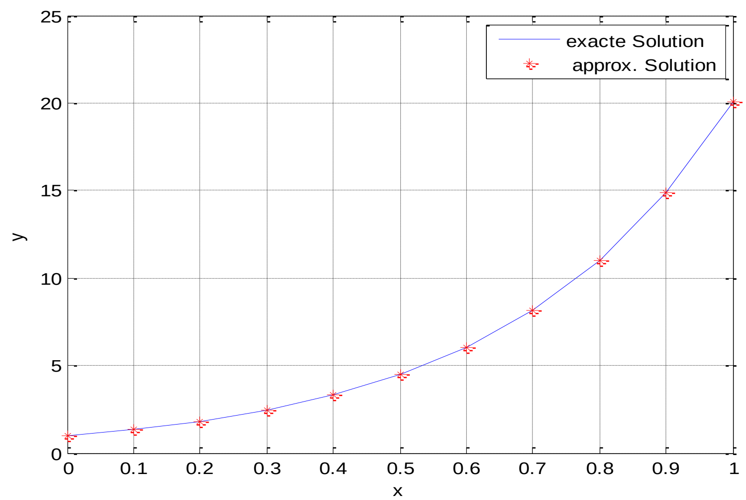

Figure 3: Graph for example 3

Example 4. *Let us consider the linear integro-differential equation of Fredholm.*

$$u'(x) = 3 + 6x + \int_0^1 xt\, u(t)dt$$

With the initial condition $u(0) = 0$

Where the function $f(x)$ *is chosen such that the exact solution is given by*

$$u(x) = 3x + 4x^2$$

The approximate solution $u_n(x)$ of $u(x)$ is obtained by the Galerkin-Genocchi polynomial method.

Table 4: We Present the Exact and Approximate Solutions of the Equation in Example 4 in Some Arbitrary Points, the Error for N=9

x	***exact solution*** u	***approximate solution*** u_n	***Error***
0.0	0.00000	0.00000	0.000000e+00
0.1	0.34000	0.34000	0.000000e+00
0.2	0.76000	0.76000	0.000000e+00
0.3	1.26000	1.26000	0.000000e+00
0.4	1.84000	1.84000	0.000000e+00
0.5	2.50000	2.50000	0.000000e+00
0.6	3.24000	3.24000	0.000000e+00
0.7	4.06000	4.06000	0.000000e+00
0.8	4.96000	4.96000	0.000000e+00
0.9	5.94000	5.94000	0.000000e+00
1	7.00000	7.00000	0.000000e+00

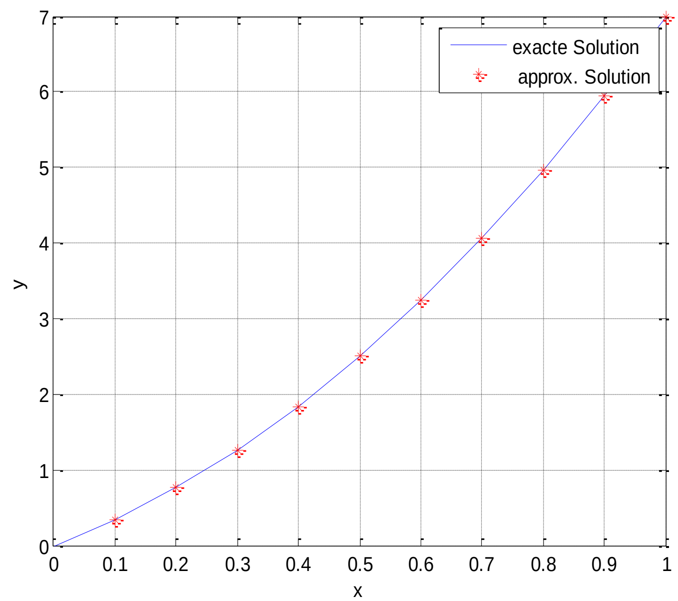

Figure 4: Graph for example 4

Example 5. *Consider linear Volterra integro-differential equation of second kind*

$$u''(x) = x + \int_0^x (x-t)u(t)\,dt$$

With the initial condition $u(0) = 0, u'(0) = 1$

Where the function $f(x)$ *is chosen such that the exact solution is given by*

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$$u(x) = \sinh(x)$$

The approximate solution $u_n(x)$ of $u(x)$ is obtained by the Galerkin-Genocchi polynomial method.

Table 5: We Present the Exact and Approximate Solutions of the Equation in Example 5 in Some Arbitrary Points, the error for N=10

x	***exact solution u***	***approximate solution u_n***	***Error***
0.0	0.0000	0.0000	0.000000e+00
0.1	0.1002	0.1002	3.234746e-12
0.2	0.2013	0.2013	5.589917e-12
0.3	0.3045	0.3045	9.178436e-12
0.4	0.4108	0.4108	1.205841e-11
0.5	0.5211	0.5211	1.446709e-11
0.6	0.6367	0.6367	1.795752e-11
0.7	0.7586	0.7586	2.116352e-11
0.8	0.8881	0.8881	2.350942e-11
0.9	1.0265	1.0265	2.713829e-11
1	1.1752	1.1752	2.997913e-11

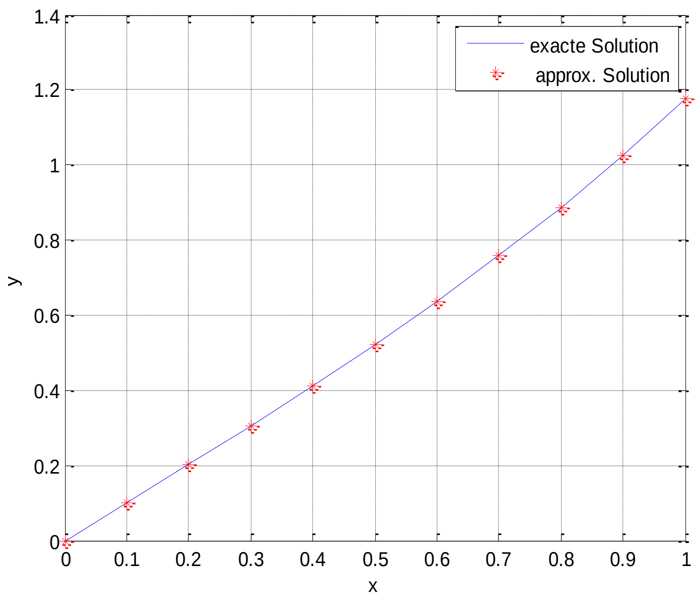

Figure 5: Graph for example 5

Example 6. *Consider the integro-differential equation*

$$u'(x) = -\cos(2\pi x) - 2\pi\sin(2\pi x) - \frac{1}{2}\sin(4\pi x) + \int_0^1 \sin(4\pi x + 2\pi t)\,u(t)\,dt$$

With the initial condition $u(0) = 1$,

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Where the function $f(x)$ *is chosen such that the exact solution is given by*

$$u(x) = \cos(2\pi x)$$

The approximate solution $u_n(x)$ *of* $u(x)$ *is obtained by the Galerkin-Genocchi polynomial method.*

Table 6: We Present the Exact and Approximate Solutions of the Equation in Example 6 in Some Arbitrary Points, the Error for N=9

x	***exact solution*** u	***approximate solution*** u_n	***Error***
0.0	1.0000	1.0000	7.227990e-08
0.1	0.8090	0.8090	1.829569e-05
0.2	0.3090	0.3090	9.117870e-05
0.3	-0.3090	-0.3090	1.261507e-05
0.4	-0.8090	-0.8090	4.260336e-05
0.5	-1.0000	-1.0000	1.261118e-04
0.6	-0.8090	-0.8090	4.352714e-05
0.7	-0.3090	-0.3090	1.050012e-05
0.8	0.3090	0.3090	8.549721e-05
0.9	0.8090	0.8090	2.149635e-05
1	1.0000	1.0000	1.244871e-05

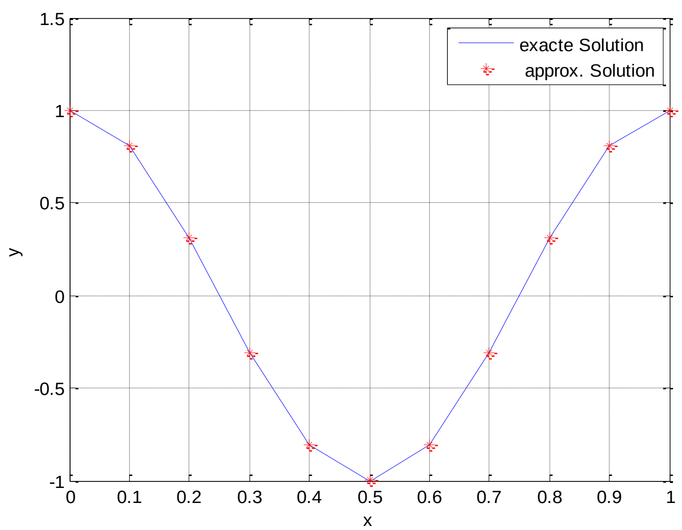

Figure 6: Graph for example 6

5. CONCLUSION

This article deals with the numerical solution of first order Fredholm and Volterra integro-differential equations of the second kind, using the Galerkin method by means of Genocchi polynomials. This technique was tested on six examples, and the results were satisfactory and the method was quite effective. In addition, this method can be applied

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to high order Fredholm and Volterra integro-differential equations of the second kind, where the Matlab program is used to obtain approximate solutions.

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