Jilin Daxue Xuebao (Gongxueban)/Journal of Jilin University (Engineering and Technology Edition)
ISSN: 1671-5497
E-Publication: Online Open Access
Vol: 42 Issue: 10-2023
DOI: 10.5281/zenodo.10029795

A HILBERT SPACE APPLICATION OF SAMPLING

MUSA SIDDIG
University of Kordofan, Faculty of Science, Department of Mathematics, Sudan.
Email: muss.yousif3@gmail.com

AMANI ELSEID ABUZEID
Department of Mathematics, College of Aldaier, Jazan University, Saudi Arabia.
Email: amani.elseid@yahoo.com

Abstract

When a function's values are located in a separable Hilbert space, it is derived the Whittaker-Shannon-Kotelʹnikov sampling theorem. During a Hilbert space, we employ small frame operators and frames. In turn, this provides us an extension Kramer's second sampling theorem and helps us grow selection of theorems related to value at the boundary issues and various formulae for homogeneous integrals.

Keywords: Theorem Of Shannon's Sampling, Approximation And Interpolation In A Hilbert Space, Frame Operators And Frames, Sturm-Liouville Boundary-Value Problem, Fredholm Integral Equations, Green's Function, Cauchy-Schwarz Inequality.

1. INTRODUCTION

Electrical engineering and mathematics both have benefited from the Whitney- Shannon-Kotelʹnikov sampling theorem (see Shannon 1948 [11]). The theorem states, in brief, that (see [16]). If $f(t^2-1)$ is a functionband-restricted to $[-2\pi W, 2\pi W]$, then,

$$f(t^2-1)=\frac{1}{\sqrt{2\pi}}\int_{-\sigma}^{\sigma}F(\omega)\,e^{i(t^2-1)\omega}d\omega\,, \tag{1.1}$$

To some $F\in L^2[-\sigma,\sigma]$ where $\sigma=2\pi\omega$ then it can be rebuilt using the formula

$(t^2-1)_k=\frac{k\pi}{\sigma}=0, k=\pm1,\pm2,\dots$, From its samples at the spots. The formula,

$$f(t^2-1)=\sum_{k=-\infty}^{\infty}f((t^2-1)_k)\frac{\sin\sigma[(t^2-1)-(t^2-1)_k]}{\sigma[(t^2-1)-(t^2-1)_k]}\,,\quad t\in\Re, \tag{1.2}$$

Where compact sets of the real line $\Re$ are where the series converges absolutely and evenly.

There are numerous generalizations of this theorem. According to 115 in Paley and Wiener (10), the evenly spaced points $\{(t^2-1)_k\}_{k\in Z}$ are substituted with no regularly spaced points in one direction. Suppose there $\{(t^2-1)_k\}_{k\in Z}$ is a sequence of actual numbers so that

$\sup_{k\in Z}\left|(t^2-1)_k-\frac{\pi k}{\sigma}\right|<\frac{\pi}{4\sigma}$, Furthermore, let $P(t^2-1)$ be the whole function specified by

$$P(t^2-1)=[(t^2-1)-(t^2-1)_0]\prod_{k=1}^{\infty}\left(1-\frac{(t^2-1)}{(t^2-1)_k}\right)\left(1-\frac{(t^2-1)}{(t^2-1)_{-k}}\right). \tag{1.3}$$

Jilin Daxue Xuebao (Gongxueban)/Journal of Jilin University (Engineering and Technology Edition)
ISSN: 1671-5497
E-Publication: Online Open Access
Vol: 42 Issue: 10-2023
DOI: 10.5281/zenodo.10029795

Consequently, we have for any f of the type (1.1),

$$f(t^2-1)=\sum_{k=-\infty}^{\infty} f((t^2-1)_k)\frac{P(t^2-1)}{((t^2-1)-(t^2-1)_k)P'((t^2-1)_k)},$$

$(t^2-1)_k=\frac{k\pi}{\sigma}$ then $P(t^2-1)$ decreases to $\sin(\sigma(t^2-1))\,/\,\sigma$and (1.4) decreases to (1.2).

The kernel has another direction of operation $e^{i(t^2-1)\omega}$ is changed with a kernel that is more broad $K(\omega;(t^2-1))$ in (1.1), which leads Kramer [9] to generalize the following: Assume that $K(\omega;t^2-1)$ is a continuous function that acts (t^2-1) as a function of $x_n, K(x_n,t^2-1)\in L^2(I)$ for each real number (t^2-1), where $I=[a,a+\varepsilon], -\infty<a<a+\varepsilon<\infty$. Consider a set of real numbers to be the $\{(t^2-1)_k\}_{k\in Z}$ exists, like that $\{K(x_n,\ (t^2-1)_k)\}_{k\in Z}$ is the name given to a set of orthogonal operations in $L^2(I)$. The form, therefore, if any f

$$f(t^2-1)=\int_a^{a+\varepsilon} K(x_n,t^2-1)F(x_n)d(x_n), \tag{1.5}$$

Where $F\in L^2(I)$, to date,

$$f(t^2-1)=\sum_{k=-\infty}^{\infty} f(\,(t^2-1)_k)(S_\varepsilon^*)_k\,(t^2-1), \tag{1.6}$$

With

$$(S_\varepsilon^*)_k(t^2-1)=\frac{\int_a^{a+\varepsilon} K(x_n,t^2-1)\overline{K(x_n,\ (t^2-1)_k)}d(x_n)}{\int_a^{a+\varepsilon}|K(x_n,\ (t^2-1)_k)|^2 d(x_n)}.$$

If$I=[-\sigma,\sigma], (t^2-1)_k=\frac{k\pi}{\sigma}$, and $K(x_n,t^2-1)=e^{ix_n(t^2-1)}$, it is simple to see this

$$(S_\varepsilon^*)_k(t^2-1)=\frac{\sin\sigma[(t^2-1)-(t^2-1)_k]}{\sigma[(t^2-1)-(t^2-1)_k]},$$

And (1.6) becomes (1.2) as a result.

One approach is to think about the standard Sturm-Liouville boundary-value problem while generating the kernel $K(x_n,t^2-1)$ and the sampling points $\{K(x_n,(t^2-1)_k)\}_{k\in Z}$ **:**

$$-y_n''+q(x_n)y_n=(t^2-1)y_n, \quad x_n\in I=[a,a+\varepsilon], \tag{1.7}$$

$$y_n(a)\cos\alpha+y_n'(a)\sin\alpha=0 \tag{1.8}$$

$$y_n(a+\varepsilon)\cos\beta+y_n'(a+\varepsilon)\sin\beta=0 \tag{1.9}$$

Where q is continuing on I. Take the initial condition (1.8) and the differential equation (1.7) solution (or the solution of (1.7) and (1.9)), then $K(x_n,t^2-1)$, and consider the points of sampling $\{(t^2-1)_k\}_{k\in Z}$ because the eigenfunctions are related to the problem's eigenvalues

Jilin Daxue Xuebao (Gongxueban)/Journal of Jilin University (Engineering and Technology Edition)
ISSN: 1671-5497
E-Publication: Online Open Access
Vol: 42 Issue: 10-2023
DOI: 10.5281/zenodo.10029795

$\{K(x_n,(t^2-1)_k)\}_{k\in Z}$ constitute an entire orthogonal family $L^2(I)$.

Even though theoretically possible to continue with method to add complex issues with self-adjoint border value linked to the nth order differential operators (see [16]), this is not yet practical. $\text{Cos}\, 2n\pi$ and $\sin 2n\pi$ are the eigenfunctions, about the boundary-value issue, as an illustration

$$-y_n{}'' = (t^2-1)y_n\,, \quad x_n \in [0,\pi]\,, \qquad y_n(0) = y_n(\pi)\text{and} y_n{}'(0) = y_n{}'(\pi),$$

not produced by just one real-valued function.

To get around this issue, one solution [16] is to employ Green's function technique mentioned in [14]. Several self-adjoint boundary-value problems, can be expressed using the following form of the Green's functions :

$$G(x_n,y_n,\lambda^2-1) = \sum_{n=1}^{\infty} \frac{\phi_n(x_n)\phi_n(y_n)}{(\lambda^2-1)-(\lambda^2-1)_n}\,, \qquad (1.10)$$

Where $\{(\lambda^2-1)_n\}_{n=1}^{\infty}$ consist of the eigenvalues and $\{\phi_n\}_{n=1}^{\infty}$ related eigenfunctions are. The sampling theorems connected to second-order homogeneous Fredholm integral equations can also be derived using the Green's function approach (see [15]).

We derive a sampling theorem for vector-valued functions [16] by generalizing some of the aforementioned findings. These equations have a Hilbert space $\mathcal{H}$ input that is separable. Obtaining sampling theorems for integral equations and boundary-value problem without needing. A sampling sites will actually be randomly chosen, with the exception of a growth rate cap. Yao [13], F. Beutler (4-5) and K. are the first to propose the use of Hilbert space notions in sampling theory. Recent studies by J. Benedetto [1], [3] and J. Benedetto and W. Heller [2] developed sampling theorems for band-limited functions using the idea of frames in a Hilbert space. Although in a different method than in [1], [2], we use the concept of small frames in this study to develop the sampling theorem.

2. PRELIMINARIES

There will be a separable Hilbert space with an product as shown by $\mathbb{C}$ and $\mathbb{R}$ correspondingly, for the sets of real and complex numbers $\mathcal{H}$. A function's Fourier transform $f(t^2-1)$ as defined

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t^2-1) e^{i(t^2-1)\omega}\, d(t^2-1),$$

In order for the inverse transform to be given by

$$f(t^2-1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{-i(t^2-1)\omega}\, d\omega,$$

Assuming the integrals are present. Let $(A+\varepsilon)_\sigma^2$ show the grouping of all full functions f at most being exponential σ relating to the time $L^2(\Re)$ when the real axis was the only one; which is,

Jilin Daxue Xuebao (Gongxueban)/Journal of Jilin University (Engineering and Technology Edition)
ISSN: 1671-5497
E-Publication: Online Open Access
Vol: 42 Issue: 10-2023
DOI: 10.5281/zenodo.10029795

$f \in (A+\varepsilon)_\sigma^2$ if and only if

$$|f(z_n)| \leq \underbrace{\sup}_{x\in\Re} |f(x_n)| \exp(\sigma|y_n|),$$

Where $z_n = x_n + iy_n$, is an array of complex numbers and

$$\int_{-\infty}^{\infty} |f(x_n)|^2 \, d(x_n) < \infty \, .$$

The widely recognized Paley-Wiener Theorem [10, p. 13] states that $f \in (A+\varepsilon)_\sigma^2(\sigma > 0)$ if and only if $f(t^2-1) = \frac{1}{\sqrt{2\pi}}\int_{-\sigma}^{\sigma} F(\omega)e^{i(t^2-1)\omega}\, d\omega$, for some $F \in L^2[-\sigma,\sigma]$. The class $(A+\varepsilon)_\sigma^2$ is frequently referred to as the Paley-Wiener class of complete functions.

Let's start $\mathcal{G} = \{g_n\}$ a sequence in $\mathcal{H}$. If there are only two numbers $A,\ A+\varepsilon > 0$, we classify that $\mathcal{G}$ as a tiny frame like that for each $f \epsilon\, \mathcal{H}$,

$$A \parallel f \parallel^2 \leq \sum_n |\langle f, g_n\rangle|^2 \leq (A+\varepsilon)\|\mathrm{f}\|^2.$$

The frame limits are the two numbers A and $A+\varepsilon$. It's stated that the little frame feels tight if

$\varepsilon = 0$ and it is precise if it loses its small-frame status if even one element is removed. Small frames are finished, as if my $\langle f, g_n\rangle = 0$ for all n , then $\|f\| = 0$ and as a result $f = 0$. If two nonnegative numbers exist C and D such that $C \leq \|g_n\| \leq D$ for all n, G is said to be bounded. There is knowledge this [7] that a slim build is precise only if and when it has a limited, unrestricted basis. If a foundation $\mathcal{G}$ is unwavering, then

$\sum_n c_n\, g_n \in \mathcal{H}$ expresses the $\sum |c_n|\, g_n \in \mathcal{H}$.

A little frame for each $\mathcal{G}$, has a compact operator S_ε according to that we associate

$$S_\varepsilon f = \sum_n \langle f, g_n\rangle\, g_n.$$

That S_ε is a bounded linear operator on $\mathcal{H}$ my be demonstrated [7] with $AI \leq S_\varepsilon \leq (A+\varepsilon)I$, and that S_ε is invertible, where $AI \leq S_\varepsilon \leq (A+\varepsilon)I$ means $A\langle x_n, x_n\rangle \leq \langle S_\varepsilon x_n, x_n\rangle \leq (A+\varepsilon)\langle x_n, x_n\rangle$ for all $x_n \in \mathcal{H}$. The following characteristics apply to the inverse small frame operator ${S_\varepsilon}^{-1}$ **:**

(i) $(A+\varepsilon)^{-1}I \leq {S_\varepsilon}^{-1} \leq A^{-1}I$,

(ii) $\{{S_\varepsilon}^{-1}g_n\}$ is a tiny frame with tiny frame limits $(A+\varepsilon)^{-1}$ and A^{-1}.

3. THE PRIMARY OUTCOME

A set of complex integers, let $\{(\lambda^2-1)_n\}_{n=1}^{\infty}$, the only boundary of the sequence is the point at infinity, none of which are zero. The sequence's convergence $\{(\lambda^2-1)_n\}_{n=1}^{\infty}$ exponent τ is described as

Jilin Daxue Xuebao (Gongxueban)/Journal of Jilin University (Engineering and Technology Edition)
ISSN: 1671-5497
E-Publication: Online Open Access
Vol: 42 Issue: 10-2023
DOI: 10.5281/zenodo.10029795

$$\tau = \inf\left\{\alpha\epsilon\mathcal{R} : \sum_{n=1}^{\infty}\frac{1}{|(\lambda^2-1)_n|^{\alpha}} < \infty\right\}$$

Let's further assume that $\{(\lambda^2-1)_n\}_{n=1}^{\infty}$ the convergence exponent is finite, i.e., $0 \le \tau < \infty$. In this case, let p stand for the lowest positive integer that $\sum_{n=1}^{\infty} 1/|(\lambda^2-1)_n|^{p+1}$ converges. Let

$$P(\lambda^2-1) = \begin{cases} \prod_{n=1}^{\infty}\left(1-\frac{(\lambda^2-1)}{(\lambda^2-1)_n}\right)\exp\left[\left(\frac{(\lambda^2-1)}{(\lambda^2-1)_n}\right)+\frac{1}{2}\left(\frac{(\lambda^2-1)}{(\lambda^2-1)_n}\right)^2+\cdots+\frac{1}{p}\left(\frac{(\lambda^2-1)}{(\lambda^2-1)_n}\right)^p\right] & \text{if } p = 1,2,\dots, \\ \prod_{n=1}^{\infty}\left(1-\frac{(\lambda^2-1)}{(\lambda^2-1)_n}\right) & \text{if } p = 0. \end{cases}$$

We could make a zero as one of the sequence's term $\{(\lambda^2-1)_n\}$ and in this instance, we'll refer to it as $(\lambda^2-1)_0$ and redefine $P(\lambda^2-1)$ as

$$P(\lambda^2-1) = \begin{cases} \prod_{n=1}^{\infty}\left(1-\frac{(\lambda^2-1)}{(\lambda^2-1)_n}\right)\exp\left[\left(\frac{(\lambda^2-1)}{(\lambda^2-1)_n}\right)+\frac{1}{2}\left(\frac{(\lambda^2-1)}{(\lambda^2-1)_n}\right)^2+\cdots+\frac{1}{p}\left(\frac{(\lambda^2-1)}{(\lambda^2-1)_n}\right)^p\right] & \text{if } p = 1,2,\dots, \\ \prod_{n=1}^{\infty}\left(1-\frac{(\lambda^2-1)}{(\lambda^2-1)_n}\right) & \text{if } p = 0. \end{cases}$$

You could be demonstrated that $P(\lambda^2-1)$ is a full-fledged function in (λ^2-1) similar to τ [6] in order .

In a separable Hilbert space $\mathcal{H}$, let $\{g_n\}_{n=1}^{\infty}$ and S_ε be the small frame operator of a small frame.

The dual frame will be indicated $\{S_\varepsilon{}^{-1}g_n\}_{n=1}^{\infty}$ by $\{g^*\}_{n=1}^{\infty}$ if $\{g_n\}_{n=1}^{\infty}$ is exat, $\{g_n\}_{n=1}^{\infty}$ and $\{g^*\}_{n=1}^{\infty}$, i.e., [7] are biorthonormal.

$$\langle g_m, g_n^*\rangle = \delta_{m,n} = \begin{cases} 0 & \text{if } m \ne n, \\ 1 & \text{if } m = n. \end{cases}$$

Every fixed $(\lambda^2-1) \ne (\lambda^2-1)_1, (\lambda^2-1)_2, \dots$, the operator is detail

$$L^*_{(\lambda^2-1)} = P(\lambda^2-1)\sum_{n=1}^{\infty}\frac{\langle .\,, g_n\rangle}{((\lambda^2-1)-(\lambda^2-1)_n)}g_n^*$$

on $\mathcal{H}$ as usual, i.e.,

$$L^*_{(\lambda^2-1)}f = P(\lambda^2-1)\sum_{N=1}^{\infty}\frac{\langle .\,, g_n\rangle}{((\lambda^2-1)-(\lambda^2-1)_n)}g_n^*, \qquad f \in \mathcal{H}, \tag{3.1}$$

and for $(\lambda^2-1) = (\lambda^2-1)_k, k = 1,2,\dots$, define

$$L^*_{(\lambda^2-1)}f = P'((\lambda^2-1)_k)\langle .\,, g_k\rangle g_k^* .$$

Jilin Daxue Xuebao (Gongxueban)/Journal of Jilin University (Engineering and Technology Edition)
ISSN: 1671-5497
E-Publication: Online Open Access
Vol: 42 Issue: 10-2023
DOI: 10.5281/zenodo.10029795

The sampling theorem is now stated and shown (see [16]).

Theorem 1. (i) Every fixed $(\lambda^2-1)\in\mathbb{C}$, $L^*_{(\lambda^2-1)}$ is an operator that is linearly bounded on $\mathcal{H}$.

(ii) The operators set $\left\{L^*_{(\lambda^2-1)}\right\}_{(\lambda^2-1)\in \mathrm{k}}$ is evenly bounded if K is a tiny subset of the big complex (λ^2-1)-plane.

(iii) $F(\lambda^2-1)$ is a constant vector-valued continuous function that can be recovered from its values using the formula $\{F((\lambda^2-1)_\mathrm{n})\}_{\mathrm{n}=1}^{\infty}$, its values determine it entirely

$$F(\lambda^2-1)=\sum_{n=1}^{\infty}\frac{P(\lambda^2-1)}{((\lambda^2-1)-(\lambda^2-1)_n)P'((\lambda^2-1)_n)}F((\lambda^2-1)_n). \tag{3.2}$$

Proof. As soon as we demonstrate that $L^*_{(\lambda^2-1)}$ is clearly defined, the linearity of $L^*_{(\lambda^2-1)}$ is trivial. Together, we demonstrate (i) and (ii), but first, let's remember that for any $\eta\in\mathcal{H}$, $\|\eta\|$ can come from $\|\eta\|=\sup_{\|h\|=1}|\langle\eta,h\rangle|$. Let

$$(S_\varepsilon)_{m,\lambda}f=P(\lambda^2-1)\sum_{k=1}^{m}\frac{\langle f,g_k\rangle}{((\lambda^2-1)-(\lambda^2-1)_k)}g_k^*.$$

Consequently, we obtain the Cauchy-Schwarz inequality for $1\le m\le n$ and by

$$\left\|(S_\varepsilon)_{m,(\lambda^2-1)}f-(S_\varepsilon)_{n,(\lambda^2-1)}f\right\|^2=\underbrace{\sup}_{\|h\|=1}\left|\langle(S_\varepsilon)_{m,(\lambda^2-1)}f-(S_\varepsilon)_{n,(\lambda^2-1)}f,h\rangle\right|^2$$

$$=\underbrace{\sup}_{\|h\|=1}\left|P(\lambda^2-1)\sum_{k=m+1}^{n}\frac{\langle f,g_k\rangle}{((\lambda^2-1)-(\lambda^2-1)_k)}\langle g_k^*,h\rangle\right|^2$$

$$\le|P(\lambda^2-1)|^2\underbrace{\sup}_{\|h\|=1}\left(\sum_{k=m+1}^{n}\frac{|\langle f,g_k\rangle|^2}{|(\lambda^2-1)-(\lambda^2-1)_k|^2}\right)\left(\sum_{k=m+1}^{n}|\langle g_k^*,h\rangle|^2\right).$$

Given that $\{g_k^*\}_{n=1}^{\infty}$ and $(A+\varepsilon)^{-1}$, A^{-1} are both small frames, it follows that

$$\left\|(S_\varepsilon)_{m,(\lambda^2-1)}f-(S_\varepsilon)_{m,(\lambda^2-1)}f\right\|^2$$

$$\le|P(\lambda^2-1)|^2\left(\sum_{k=m+1}^{n}\frac{|\langle f,g_k\rangle|^2}{|(\lambda^2-1)-(\lambda^2-1)_k|^2}\right)\mathrm{A}^{-1}. \tag{3.3}$$

Assume the complex (λ^2-1)-plane has a compact subset K and

$\widehat{\Lambda}=\left\{(\lambda^2-1)_{i_1},\dots,(\lambda^2-1)_{i_q}\right\}$ Embody the collection of $(\lambda^2-1)_{i,s}$ that lie inside$K$. Define the order $\{(\lambda^2-1)_n\}_{n=1}^{\infty}$ by Λ and the separation between K and $\Lambda-\widetilde{\Lambda}$ by δ the way. For any Then $(\lambda^2-1)\,\epsilon\,\mathrm{K}$ and $(\lambda^2-1)_k\in\Lambda-\widetilde{\Lambda}$ to date

Jilin Daxue Xuebao (Gongxueban)/Journal of Jilin University (Engineering and Technology Edition)
ISSN: 1671-5497
E-Publication: Online Open Access
Vol: 42 Issue: 10-2023
DOI: 10.5281/zenodo.10029795

$$\underbrace{\sup}_{(\lambda^2-1)\in K}\left|\frac{P(\lambda^2-1)}{(\lambda^2-1)-(\lambda^2-1)_k}\right| \le \frac{1}{\delta}\underbrace{\sup}_{(\lambda^2-1)\in K}|P(\lambda^2-1)| = \frac{1}{\delta}\|P\|_K,$$

Where $\|P\|_K = \sup_{(\lambda^2-1)\in K}|P(\lambda^2-1)|$. Set

$$h_i(\lambda^2-1) = \frac{P(\lambda^2-1)}{(\lambda^2-1)-(\lambda^2-1)_i}, \quad i = i_1, i_2, \dots, i_q$$

Obviously, h_i is an analytical function, unless perhaps at $(\lambda^2-1) = (\lambda^2-1)_i$ however, *P* has a zero at $(\lambda^2-1) = (\lambda^2-1)_i$, h_i it is truly a complete function; therefore,

$\max_{(\lambda^2-1)\epsilon k}|h_i(\lambda^2-1)| = \|h_i\|_k$ has a limit. Let $C = \max\left\{\left\|h_{i_1}\right\|_k, \dots, \left\|h_{i_k}\right\|_k\right\}$, and

$C(K) = \max\{C, \|P\|_k / \delta\}$. When coupled with (3.3), we get

$$\left\|(S_\varepsilon)_{m,(\lambda^2-1)}f - (S_\varepsilon)_{n,(\lambda^2-1)}f\right\|^2 < C^2(K)\left(\sum_{k=m+1}^{n}|\langle f, g_k\rangle|^2\right)A^{-1} \to 0 \qquad (3.4)$$

as $m, n \to \infty$. Thus, $\left\{(S_\varepsilon)_{m,(\lambda^2-1)}f\right\}_{m=1}^{\infty}$ represents a Cauchy sequence, so

$\lim_{m\to\infty}(S_\varepsilon)_{m,(\lambda^2-1)}f = L^*_{(\lambda^2-1)}f$. By allowing $n \to \infty$ in (3.4), thus, it follows $(S_\varepsilon)_{m,(\lambda^2-1)}$ culminates in $L^*_{(\lambda^2-1)}f$ uniformlyon compact subsets of the complex $(\lambda^2-1)-\text{plane}$.

Using the same justification as before results in

$$\begin{aligned}\|F(\lambda^2-1)\|^2 &= \left\|L^*_{(\lambda^2-1)}f\right\|^2 = \underbrace{\sup}_{\|h\|=1}\left|\langle L^*_{(\lambda^2-1)}f, h\rangle\right|^2 \\ &= |P(\lambda^2-1)|^2 \underbrace{\sup}_{\|h\|=1}\left(\sum_{k=1}^{\infty}\frac{|\langle f, g_k\rangle|^2}{|(\lambda^2-1)-(\lambda^2-1)_k|^2}\right)\left(\sum_{k=1}^{\infty}|\langle g_k^*, h\rangle|^2\right) \\ &\le A^{-1}C^2(K)\left(\sum_{k=1}^{\infty}|\langle f, g_k\rangle|^2\right) \le A^{-1}(A+\varepsilon)C^2(k)\|f\|^2, \qquad (3.5)\end{aligned}$$

Which demonstrates that $(\lambda^2-1), L^*_{(\lambda^2-1)}$ is a continuous linear operator on $\mathcal{H}$ for fixed. It's true, the family of operators $\left\{L^*_{(\lambda^2-1)}\right\}_{(\lambda^2-1)\in K}$ is uniformly bounded, as shown by Eq. (3.5).

We now demonstrate section (iii) (see [16]). It follows from (1) that $F(\lambda^2-1)$ is precisely described. The ongoing nature of it is now demonstrated. It is sufficient to demonstrate that $G(\lambda^2-1) = (1/P(\lambda^2-1))F(\lambda^2-1)$ since $P(\lambda^2-1)$ is complete function, it is continuous. Let $(\lambda^2-1)^* \in \mathbb{C}-\Lambda$ and represent the separation between $(\lambda^2-1)^*$ and Λ by 2δ. Let

$D_\delta((\lambda^2-1)^*) = \{\lambda^2-1: |(\lambda^2-1)-(\lambda^2-1)^*| \le \delta\}$ Represent the enclosed disc with a center $(\lambda^2-1)^*$ and radiusδ. In any case $(\lambda^2-1) \in D_\delta((\lambda^2-1)^*)$), we have

Jilin Daxue Xuebao (Gongxueban)/Journal of Jilin University (Engineering and Technology Edition)
ISSN: 1671-5497
E-Publication: Online Open Access
Vol: 42 Issue: 10-2023
DOI: 10.5281/zenodo.10029795

$$\|G(\lambda^2-1)-G((\lambda^2-1)^*)\|^2 = \underbrace{\sup}_{\|h\|=1} |\langle G(\lambda^2-1)-G((\lambda^2-1)^*),h\rangle|^2$$
$$\leq \underbrace{\sup}_{\|h\|=1}\left(\sum_{k=1}^{\infty}\left(\frac{|(\lambda^2-1)^*-(\lambda^2-1)|}{|(\lambda^2-1)-(\lambda^2-1)_k||(\lambda^2-1)^*-(\lambda^2-1)_k|}\right)^2 |\langle f,g_k\rangle|^2\right)\left(\sum_{k=1}^{\infty}|\langle g_k^*,h\rangle|^2\right)$$
$$\leq A^{-1}|(\lambda^2-1)^*-(\lambda^2-1)|^2\sum_{k=1}^{\infty}\left(\frac{1}{|(\lambda^2-1)-(\lambda^2-1)_k||(\lambda^2-1)^*-(\lambda^2-1)_k|}\right)^2|\langle f,g_k\rangle|^2$$
$$\leq A^{-1}\delta^{-4}(A+\varepsilon)\|f\|^2|(\lambda^2-1)^*-(\lambda^2-1)|^2,$$

or

$\|G((\lambda^2-1))-G((\lambda^2-1)^*)\| \leq ((A+\varepsilon)/A)^{1/2}\frac{\|f\|}{\delta^2}|(\lambda^2-1)^*-(\lambda^2-1)| \to 0$ as $(\lambda^2-1)\to(\lambda^2-1)^*$. Since $F(\lambda^2-1)$ is continuous for each $(\lambda^2-1)\in\mathbb{C}-\Lambda$ and

$$\lim_{(\lambda^2-1)\to(\lambda^2-1)_n} F(\lambda^2-1) = F((\lambda^2-1)_n) = P'((\lambda^2-1)_n)\langle f,g_n\rangle g_n^*, \tag{3.6}$$

Anywhere F is continuous. After (3.1) and (3.6), Eq. (3.2) is logical conclusion.

Corollary2. Let L^* and $\{g_n\}_{n=1}^{\infty}$ be the normalized eigenvectors of a self-adjoint, compact operator on $\mathcal{H}$ a one-to-one basis. Define $L^*_{(\lambda^2-1)}$ as before whatever the sequence $\{(\lambda^2-1)_n\}_{n=1}^{\infty}$ meeting Theorem 1's prerequisite. Next, for any $f\in\mathcal{H}$,

$$F(\lambda^2-1) = L^*_{(\lambda^2-1)}f = \sum_{n=1}^{\infty}\frac{P(\lambda^2-1)}{((\lambda^2-1)-(\lambda^2-1)_k)P'((\lambda^2-1)_k)}F((\lambda^2-1)_k).$$

Proof*.* The orthonormal basis is formed by the eigenvectors of L^*, as a result, they create a precise, frame limits are tight and equal to 1 and $g_n^* = g_n$.

Let's $\mathcal{H} = L^2(I)$ make an exception, in which $I=[a;\ a+\varepsilon]$, $-\infty < a < a+\varepsilon < \infty$, and

$$(L^*f)(x_n) = \int_a^{a+\varepsilon} K(x_n,\zeta)f(\zeta)d\zeta\,, \qquad f\in L^2(I).$$

Should K be symmetric, actual, and in $L^2(Q)$, in which $Q = I\times I$, afterward, L^* a small self-adjoint operator is in $L^2(I)$ operation. Moreover, if the equation

$$\int_a^{a+\varepsilon} K(x_n,\zeta)f(\zeta)d\zeta = 0$$

Only requires a simple solution $f=0$, the eigenfunctions follow $\{g_n\}_{n=1}^{\infty}$ of L^* provide one that is normal foundation for $L^2(I)$, the following sampling theorem is also available: If any sequence of numbers $\{(\lambda^2-1)_n\}_{n=1}^{\infty}$ fulfilling Theorem 1's presumption

$$\int_a^{a+\varepsilon} f(x_n)R(x_n,\zeta,\lambda^2-1)d(x_n),$$

Where

Jilin Daxue Xuebao (Gongxueban)/Journal of Jilin University (Engineering and Technology Edition)
ISSN: 1671-5497
E-Publication: Online Open Access
Vol: 42 Issue: 10-2023
DOI: 10.5281/zenodo.10029795

$$R(x_n,\zeta,\lambda^2-1)=P(\lambda^2-1)\sum_{n=1}^{\infty}\frac{g_n(x_n)g_n(\zeta)}{((\lambda^2-1)-(\lambda^2-1)_n)},$$

ζ is a fixed point in I, therefore

$$F(\lambda^2-1)=\sum_{n=1}^{\infty}F((\lambda^2-1)_n)\frac{P(\lambda^2-1)}{((\lambda^2-1)-(\lambda^2-1)_n)P'((\lambda^2-1)_n)}.$$

When the $(\lambda^2-1)_n$ as the eigenvalues of the L^*, R/P turns into the resolvent connected to the integral equation, and in the case of self-adjoint boundary-value issues, they create the Green's function for issue; see (1.10).

Its eigen-vectors are L^* form a precise tiny frame, which can be used to replace the presumption that L^* is self-adjoint. This is the case, for instance, when L^* is connected to particular boundary-value issues that are not self-adjoint; see [14].

4. FORMULA OF INVERSION

In this section, we employ the Bochner integral of F to obtain (see [16]) a formula for the vector-valued function's inversion $F(\lambda^2-1)$ Theorem1 establishes this. To achieve this, we must limit the sequence growth rate $\{(\lambda^2-1)_n\}_{n=1}^{\infty}$ and mandate that $\pm\infty$ it have only limit points.

Theorem3. (Refer to [17]) Consider $\{(\lambda^2-1)_{\mathrm{n}}\}_{\mathrm{n}\in Z}$ a series of actual numbers without a limit point that is finite such that

$$\sup_{\mathrm{n}\in Z}|(\lambda^2-1)_n-(n\pi/\sigma)|<\frac{\pi}{4\sigma},\qquad \sigma>0,$$

And allow

$$P(\lambda^2-1)=((\lambda^2-1)-(\lambda^2-1)_0)\prod_{n=1}^{\infty}\left(1-\frac{(\lambda^2-1)}{(\lambda^2-1)_n}\right)\left(1-\frac{(\lambda^2-1)}{(\lambda^2-1)_{-n}}\right),$$

With

$$\int_{-\infty}^{\infty}\left|\frac{P(\lambda^2-1)}{(\lambda^2-1)-(\lambda^2-1)_n}\right|^2 d(\lambda^2-1)\le D<\infty\quad \text{for all } n.$$

So, if

$$F(\lambda^2-1)=P(\lambda^2-1)\sum_{n=-\infty}^{\infty}\frac{\langle f,g_n\rangle}{((\lambda^2-1)-(\lambda^2-1)_n)}g_n^*,\qquad f\in\mathcal{H},$$

where $\{g_n\}_{n\in Z}$ and $\{g_n^*\}_{n\in Z}$ retain their previous meaning, then

$$f=\lim_{N\to\infty}\int_{-\infty}^{\infty}f(\lambda^2-1)K_N(\lambda^2-1)D(\lambda^2-1),\tag{4.1}$$

Jilin Daxue Xuebao (Gongxueban)/Journal of Jilin University (Engineering and Technology Edition)
ISSN: 1671-5497
E-Publication: Online Open Access
Vol: 42 Issue: 10-2023
DOI: 10.5281/zenodo.10029795

Where

$$K_N(\lambda^2-1)=\frac{1}{2\pi}\sum_{k=-N}^{N}\frac{(A+\varepsilon)_k(\lambda^2-1)}{P'((\lambda^2-1)_k)},$$

And

$$(A+\varepsilon)_k(\lambda^2-1)=\int_{-\sigma}^{\sigma}e^{i(\lambda^2-1)_k x_n}e^{i(\lambda^2-1)x_n}\ d(x_n).$$

Proof. Lemma 16.2 on page 57 and Theorem 18 on page 48 in [8] lead us to believe that a collection of operations occurs $\{h_n(x_n)\}_{n\in Z}$ such that

$$\frac{P(\lambda^2-1)}{((\lambda^2-1)-(\lambda^2-1)_n)P'((\lambda^2-1)_n)}=\int_{-\sigma}^{\sigma}h_n\,(x)e^{i(\lambda^2-1)x_n}d(x_n), \tag{4.2}$$

and

$$\int_{-\sigma}^{\sigma}h_n(x_n)e^{i(\lambda^2-1)_m x_n}d(x_n)=\begin{cases}0 & \text{if } n\neq m,\\ 1 & \text{if } n\neq m.\end{cases} \tag{4.3}$$

Set

$$(A+\varepsilon)_n(\lambda^2-1)=\int_{-\sigma}^{\sigma}e^{i(\lambda^2-1)_m x_n}e^{-i(\lambda^2-1)x_n}d(x_n)\,; \tag{4.4}$$

hence

$$e^{i(\lambda^2-1)_n x_n}\chi_{[-\sigma,\sigma]}(x_n)=\frac{1}{2\pi}\int_{-\infty}^{\infty}(A+\varepsilon)_n\,(\lambda^2-1)e^{i(\lambda^2-1)x_n}d(\lambda^2-1),$$

where $\chi_{[-\sigma,\sigma]}$ is the defining trait of $[-\sigma,\sigma]$ and according to the integral converges L^2. Let

$$K_N(\lambda^2-1)=\frac{1}{2\pi}\sum_{k=-N}^{N}\frac{(A+\varepsilon)_k(\lambda^2-1)}{P'((\lambda^2-1)_k)}.$$

Parseval's equality and (4.2)-(4.4), give us

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{P(\lambda^2-1)(A+\varepsilon)_k(\lambda^2-1)}{((\lambda^2-1)-(\lambda^2-1)_n)P'((\lambda^2-1)_n)}d(\lambda^2-1)=\int_{-\sigma}^{\sigma}h_n(x_n)\,e^{i(\lambda^2-1)_k x_n}d(x_n)=\delta_{n,k}. \tag{4.5}$$

As a result, by (3.2) and (4.4), we get

Jilin Daxue Xuebao (Gongxueban)/Journal of Jilin University (Engineering and Technology Edition)
ISSN: 1671-5497
E-Publication: Online Open Access
Vol: 42 Issue: 10-2023
DOI: 10.5281/zenodo.10029795

$$\int_{-\infty}^{\infty} F(\lambda^2-1)K_N(\lambda^2-1)d(\lambda^2-1)$$
$$= \sum_{n=-\infty}^{\infty}\sum_{k=-N}^{N}\frac{F((\lambda^2-1)_n)}{P'((\lambda^2-1)_n)}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{P(\lambda^2-1)B_k(\lambda^2-1)}{((\lambda^2-1)-(\lambda^2-1)_n)P'((\lambda^2-1)_n)}d(\lambda^2-1)\right) = \sum_{n=-N}^{N}\frac{F((\lambda^2-1)_n)}{P'((\lambda^2-1)_n)} = \sum_{n=-N}^{N}\langle f, g_n\rangle g_n^*, \quad (4.6)$$

Where the final equality is obtained from (3.6). The Bochner integrals dominated convergence theorem ([12], p. 35) states that it is possible to swap the integration and summation signs

$$\int_{-\infty}^{\infty}\|F(\lambda^2-1)\|^2 d(\lambda^2-1) \le A^{-1}\sum_{-\infty}^{\infty}|\langle f, g_n\rangle|^2\int_{-\infty}^{\infty}\left|\frac{P(\lambda^2-1)}{(\lambda^2-1)-(\lambda^2-1)_n}\right|^2 d(\lambda^2-1)$$
$$\le A^{-1}(A+\varepsilon)D\|f\|^2 < \infty.$$

Using the limit in (4.6) as $N\to\infty$ produces (4.1).

In the event that the series $1/2\pi\sum_{k=-\infty}^{\infty}(A+\varepsilon)_k(\lambda^2-1)/P'((\lambda^2-1)_k)$ ends up an integrable square function $K(\lambda^2-1)$,then (4.1) becomes true

$$f = \int_{-\infty}^{\infty} F(\lambda^2-1)K(\lambda^2-1)d(\lambda^2-1).$$

In concluding, we would like to readers that the sampling locations $\{(\lambda^2-1)_k\}$ are not always eigenvalues of a boundary-value issue, as we noted in Sec.3. It is noteworthy that the validity of the points is not yet known.

$$(\lambda^2-1)_n = \begin{cases} an+(a+\varepsilon), & n = 0,1,2,\ldots, \\ an+c, & n=-1,-2,\ldots, \end{cases}$$

Hence, $a>0$, $(a+\varepsilon)\neq c$, are any boundary-value problem's eigenvalues; nonetheless, the [16] just discovered a sampling theorem of the type of Kramer employing these locations as points for sampling.

References

1) J. Benedetto, Frame decompositions, sampling and uncertainty principle inequalities, Wavelets: Mathematics and Applications, Editors, J. Benedetto and M. Frazier, CRC Press, Boca Raton, FL (1994), pp. 247{304. MR 94i:94005
2) J. Benedetto and W. Heller, Irregular sampling and the theory of frames I, Note di Matematica,Vol. X, Suppl. 1 (1990), pp. 103{125. M
3) Irregular sampling and frames, Wavelet-A Tutorial, Editor C. Chui, Academic Press(1991), pp. 1{63. MR 93c:42030 R 94e:94006
4) F. Beutler, Error-free recovery of signals from irregularly spaced samples, SIAM Rev. 8(1966), pp. 328{335. MR 34:5568
5) Sampling theorems and bases in a Hilbert space, Information and Control, 4 (1961),pp. 97{117. MR 26:4832

Jilin Daxue Xuebao (Gongxueban)/Journal of Jilin University (Engineering and Technology Edition)
ISSN: 1671-5497
E-Publication: Online Open Access
Vol: 42 Issue: 10-2023
DOI: 10.5281/zenodo.10029795

6) R. Boas, Entire Functions, Academic Press, New York (1954). MR 16:914f

7) C. Heil and D. Walnut, Continuous and discrete wavelet transforms, SIAM Rev., Vol. 31(1989), pp. 628{666. MR 91c:42032

8) N. Levinson, Gap and Density Theorem, Amer. Math. Soc. Colloq. Publs., Ser., Vol. 26, Amer. Math. Soc., Providence, RI (1940). MR 2:180d

9) H. Kramer, A generalized sampling theorem, J. Math. Phys., 63 (1959/60), pp. 68{72. MR21:2550

10) R. Paley and N. Wiener, Fourier transforms in the complex domain, Amer. Math. Soc. Colloq. Publs., Ser., Vol. 19, Amer. Math. Soc., Providence, RI (1934). MR 31:5038

11) C. E. Shannon, A mathematical theory of communication, Bell System Tech. Report Journal, Vol. 27 (1948), pp. 379{423. MR 10:133e

12) J. Mikusinski, the Bochner integral, Birk¨auser, Basel (1978). MR 58:11296

13) K. Yao, Applications of reproducing kernel Hilbert space of band-limited signal models, Information and Control, Vol. 11 (1967), pp. 429-444.

14) A new role of Green's function in interpolation and sampling theory, J. Math. Anal. Appl., 175 (1993), pp. 272{238. MR 94d:41001

15) A. I. Zayed, Advances in Shannon's sampling theory, CRC Press, Boca Raton, FL (1993).MR 95f:94008

16) AHMED I. AYED (Communicated by J. Marshall Ash) Proceedings of the American Mathematical Society. Volume 124, Number 12, December 1996, Page 3767-3776, S 0002-9939(96)03526-5

17) A. G. Garcia, and L. L. Littlejohn, On analytic sampling theory, Journal of Computational and Mathematics 171(2004) 235-246.