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NOVEL TECHNIQUES FOR CONVERGENCE OF THE YOSIDA VARIATIONAL INCLUSION INCLUDING RESOLVENT EQUATION PROBLEM

ARIFUZZAMAN
Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India.

S. S. IRFAN
Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India.

SALEHIN FERDOUS KADER
Department of Electrical and Electronic Engineering, American International University, Dhaka, Bangladesh.

ABDUR RAHMAN
Department of Mathematics, National University, Gagipur, Bangladesh.

FEEROZ BABU
Department of Mathematics, VIT Bhopal University, India.

Abstract

This work examines the issue of inclusion in a real-ordered Hilbert space, specifically focusing on the Yosida approximation operator, and XOR-operator. This topic is known as the Yosida variational inclusion problem. Our study primarily centers on examining the rapid convergence of the Yosida variational inclusion problem and the resolvent equation problem. Several algorithms have been enhanced to address both issues. We prove the existence and convergence of solutions for both problems. Two mathematical models are presented to demonstrate the efficacy of the approach.

INTRODUCTION

Hassouni and Moudafi researched a category of mixed variational inequalities involving single-valued mappings, which they referred to as variational inclusions. The variational inclusion issue can be defined as the task of identifying the points where the maximal monotone mappings have a value of zero. Various scholars have explored and concluded that variational inclusions encompass and extend the concepts of variational inequalities, equilibrium problems, optimization problems, complementarity problems, and issues related to Nash equilibrium, among others. The Yosida approximation operator, described in terms of the resolvent operator, is used to approximate the derivatives of convex functionals in Hilbert spaces. The Yosida approximation operator is commonly used to work with heat equations, wave equations, and heat flow, among other applications.

The XOR logical operations are binary operations that take two Boolean operands and return true only if the operands differ. Therefore, it will yield a false result if the two operands possess identical values. The XOR-operation can be employed to verify the simultaneous falsehood of two conditions. The XOR-operation are extensively utilized in cryptography, where they produce parity bits to check for errors and ensure fault

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tolerance. It is also used in hardware to generate pseudo-random numbers and in digital computing and linear separability applications. A way to use XOR- in cryptography is given below.

Cryptograph.

	USE OF EXCLUSIVE OR GATE	BITS	RESULT
XOR Symbol $\oplus$	$0\oplus 0=0$	Same Bit	Zero
	$1\oplus 1=0$	Same Bit	Zero
	$1\oplus 0=1$	Different Bits	One
	$0\oplus 1=1$	Different Bits	One

In this research, we focus on the Yosida inclusion problem involving the XOR-operation, and its accompanying resolvent equation problem, considering the significance of the facts above. We establish several iterative techniques for resolving both of the difficulties.

Fundamental Tools

Through this paper, we suppose that $\mathcal{K}$ is called real order Hilbert Space equipped with the usual norm $\|.\|$ and inner product $\langle . , . \rangle, C \subseteq \mathcal{K}$ is called a closed convex cone, and $2^{\mathcal{K}}$ represent the set of all non-empty subsets of $\mathcal{K}$.

Definition 2.1. The relation " $\leq$ " is called a partially ordered relation induced by the cone C, provided $a \leq b$ holds if and only if $a-b \in C$, where a and b are said to be comparable if either $a \leq b$ and $b \leq a$ .The comparable elements are represented by $a \propto b$.

Definition 2.2. In the sake of arbitrary elements $a, b \in \mathcal{K}$,consider lub$\{a,b\}$ and glb$\{a,b\}$ for the set $\{a,b\}$ exist, where lub means least upper bound which is denoted by $\vee$ and $\vee$ is called OR-operation.Again, glb means greatest lower bound which is denoted by $\wedge$ and $\wedge$ is called AND-operation.Then some binary operations are given below.

(i) $a \vee b = lub\{a,b\}$

(ii) $a \wedge b = glb\{a,b\}$

(iii) $a \oplus b = (a-b) \vee (b-a)$, where $\oplus$ be an XOR operation.

(iv) $a \odot b = (a-b) \wedge (b-a)$, where $\odot$ be an XNOR operation.

Proposition 2.3. Suppose $\oplus$ is called XOR operation and $\odot$ is called XNOR operation. Then the following holds:

(i) $a \odot a = 0$, $a \odot b = b \odot a$, $a \oplus a = 0$, $(a \oplus b) = (b \oplus a)$, $(a \odot b) = -(b \oplus a)$,

(ii) $If\ a \propto 0, then - a \oplus 0 \leq a \leq a \oplus 0$

(iii) $0 \leq a \oplus b\ , If\ a \propto b$

(iv) $If\ a \propto b, then\ a \oplus b = 0\ if\ and\ only\ if\ a = b$

(v) $\|0 \oplus 0\| = \|0\| = 0$

(vi) $\|a \oplus b\| \leq \|a-b\|$

(vii) $if\ a \propto b, then\ \|a \oplus b\| = \|a-b\|$

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Definition 2.4. Suppose, $\mathfrak{D}: \mathcal{K} \times \mathcal{K} \rightarrow 2^{\mathcal{K}}$ is a multi-valued mapping. Then

(i) $\mathfrak{D}$ is called a comparison mapping, if any $v_a \in \mathfrak{D}(.,a), a \propto v_a$ and if $a \propto b$, then for any $v_a \in \mathfrak{D}(.,a)$ and $v_b \in \mathfrak{D}(.,b)$, $v_a \propto v_b, \ \forall, a, b \in \mathcal{K}$

(ii) The comparison mapping $\mathfrak{D}$ is called α -non -ordinary difference mapping, if for each $a, b \in \mathcal{K}, v_a \in \mathfrak{D}(.,a)$ and $v_b \in \mathfrak{D}(.,b)$such that $(v_a \oplus v_b) \oplus \propto (a \oplus b) = 0$

(iii) The comparison mapping $\mathfrak{D}$ is called γ-ordered rectangular mapping, if there exists a constant $\gamma > 0$ and for each $a, b \in \mathcal{K}$,there exist $v_a \in \mathfrak{D}(.,a)$ and $v_b \in \mathfrak{D}(.,b)$ such that $\langle (v_a \odot v_b) - (a \oplus b) \rangle \geq \gamma \|a \oplus b\|^2$

(iv) $\mathfrak{D}$ is called a weak comparison mapping, if any $a, b \in \mathcal{K}$, $a \propto b$, there exist $v_a \in \mathfrak{D}(.,a)$ and $v_b \in \mathfrak{D}(.,b)$ such that $a \propto v_a$, $b \propto v_b$ and $v_a \propto v_b$

(v) $\mathfrak{D}$ is called $\zeta - weak$ ordered different comparison mapping if there exists a constant $\zeta > 0$ such that for each $a, b \in \mathcal{K}$,there exist $v_a \in \mathfrak{D}(.,a)$ and $v_b \in \mathfrak{D}(.,b)$ such that $\zeta(v_a - v_b) \propto (a - b)$

(vi) A weak comparison mapping $\mathfrak{D}$ is called (γ, ζ)-weak ordered rectangular different mapping, if $\mathfrak{D}$ is a γ-ordered rectangular and $\zeta - weak$ ordered different comparison mapping and $[\tau + \zeta \mathfrak{D}(.,.)](\mathcal{K}) = \mathcal{K}, \ \ \forall, \zeta > 0$.

Definition 2.5. Suppose, $\mathfrak{D}: \mathcal{K} \times \mathcal{K} \rightarrow 2^{\mathcal{K}}$ is a multi-valued mapping. The resolvent operator $\mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,\ a)}: \mathcal{K} \rightarrow \mathcal{K}$ is defined as $\mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,\ a)}(b) = [\tau + \zeta \mathfrak{D}(.,a)]^{-1}(b) \ \forall, a, b \in \mathcal{K}$, (1)

τ is identity mapping and $\zeta > 0$ is a constant.

Definition 2.6. The Yosida approximation operator $\mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,\ a)}: \mathcal{K} \rightarrow \mathcal{K}$ is defined as $\mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,\ a)}(b) = \frac{1}{\zeta}[\tau - R(.,a)](b)$, $\forall, a, b \in \mathcal{K}$, (2)

τ is identity mapping and $\zeta > 0$ is a constant.

Lemma 2.7. Suppose, $\mathfrak{D}: \mathcal{K} \times \mathcal{K} \rightarrow 2^{\mathcal{K}}$ is γ-ordered rectangular multi-valued mapping for

$\mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,\ a)}$, Then we get, $\left\|\mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,\ a)}(\mathrm{u}) \oplus \mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,\ a)}(v)\right\| \leq \theta \|u \oplus v\|$, (3)

Where $\theta = \frac{1}{\gamma\zeta - 1}, \zeta > \frac{1}{\gamma} \quad \forall, u, v \in \mathcal{K}$

Thus, the resolvent operator $\mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,\ a)}$ is Lipschitz-type continuous.

Lemma 2.8. Suppose, $\mathfrak{D}: \mathcal{K} \times \mathcal{K} \rightarrow 2^{\mathcal{K}}$ be (γ, ζ) weak-ordered rectangular different

Multi-valued mapping with respect to $\mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,\ a)}$, then we get,

$\left\|\mathrm{Y}_{\mathrm{I},\zeta}^{\mathfrak{D}(.,\ a)}(\mathrm{u}) \oplus \mathrm{Y}_{\mathrm{I},\zeta}^{\mathfrak{D}(.,\ a)}(v)\right\| \leq \theta' \|u \oplus v\|$, where $\theta' = \frac{\zeta}{\gamma\zeta - 1}, \zeta > \frac{1}{\gamma}$, $\forall, u, v \in \mathcal{K}$ (4)

That is, the Yosida approximation operator $\mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.\ ,a)}$ is Lipschitz-type continuous.

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Statement of the problem and Iterative algorithm.

Suppose $p:\mathcal{K} \to \mathcal{K}$ is a single-valued mapping and $\mathfrak{D} : \mathcal{K} \times \mathcal{K} \to 2^{\mathcal{K}}$ is a multi-valued mapping and $\mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.\,,\ \mathrm{a})}$ is the Yosida approximation operator. To find the value $a \in \mathcal{K}$ such that

$$0 \in \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.\,,\ \mathrm{a})}(a) \oplus \mathfrak{D}(p(a), a) \quad (5)$$

Where $\zeta > 0$ is a constant and τ is the identity mapping.

If $\mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.\,,\ \mathrm{a})}(a) = 0$ *and* $\mathfrak{D}(p(a), a) = \mathfrak{D}(a)$, then problem (5) reduces to the problem of finding

$a \in \mathcal{K}$ Such that

$0 \in \mathfrak{D}(a)$, Which is the fundamental problem of analysis that has been considered by Rockafellar.

Lemma 3.1. The Yosida variational inclusion problem (5) has a solution $a \in \mathcal{K}$ if and only if it satisfies the following equation

$$p(a) = \mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}[p(a) + \zeta \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}(a)] \quad (6)$$

Proof: Suppose $a \in \mathcal{K}$ satisfies the equation (6), Then

$$p(a) = \mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}[p(a) + \zeta \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}(a)]$$

$$\Rightarrow p(a) = [\tau + \zeta \mathfrak{D}(.\,, a)]^{-1}[p(a) + \zeta \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}(a)] \quad \text{[By (1)]}$$

$$\Rightarrow p(a)(\tau + \zeta \mathfrak{D}(.\,, a)) = [p(a) + \zeta \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}(a)]$$

$$\Rightarrow p(a) + \zeta \mathfrak{D}(p(a), a) = p(a) + \zeta \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}(a)$$

$$\Rightarrow \mathfrak{D}(p(a), a) \oplus \mathfrak{D}(p(a), a) = \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}(a) \oplus \mathfrak{D}(p(a), a) \quad [\because a \oplus a = 0]$$

$0 \in \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}(a) \oplus \mathfrak{D}(p(a), a)$ Which is the required Yosida inclusion problem (5)

Now we establish the subsequent algorithm utilizing lemma 2.7 for solving the Yosida inclusion problem (5).

Algorithm 3.2. Enumerate sequence $\{a_n\}$ by taking after the iterative method

$$p(a_{n+1}) = \mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,a_n)}[p(a_n) + \zeta \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a_n)}(a_n)] \quad \text{, for every } a_0 \in \mathcal{K} \quad (7)$$

τ is the identity mapping, and $\zeta > 0$ is a constant.

MAIN RESULT AND EXPERIMENT

Theorem 3.3. Suppose $\mathcal{K}$ is a real ordered Hilbert space, and C is a cone, including partial ordering. Let, $\mathfrak{D} : \mathcal{K} \times \mathcal{K} \to 2^{\mathcal{K}}$ is the multi-valued mapping such that $\mathfrak{D}(.\,, a)$ is γ-

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ordered rectangular and (γ,ζ)-weak ordered rectangular different mapping in the first argument. Consider $p:\mathcal{K}\to\mathcal{K}$ is a single-valued mapping such that p is Lipschitz continuous with constant ζ_p and strongly monotone with constant δ_p.Let us consider $a_{n+1}\propto a_n$, $p(a_{n+1})\propto p(a_n)$, for $n=0,1,2,\dots$ ...and the subsequent axioms are fulfilled:

$$\left\|\mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.\,,\;a_n)}(\mathrm{u})\oplus \mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.\,,\;a_{n+1})}(u)\right\|\le\mu\|a_n\oplus a_{n+1}\| \tag{8}$$

$$\left\|\mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.\,,\;a_n)}(\mathrm{u})\oplus \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.\,,\;a_{n+1})}(u)\right\|\le\mu'\|a_n\oplus a_{n+1}\| \tag{9}$$

If satisfies this condition: $\{\theta\zeta_p+\mu+\theta\zeta(\mu'+\theta')<\delta_p$ (A)

Where $\theta=\frac{1}{\gamma\zeta-1}$,$\theta'=\frac{\zeta}{\gamma\zeta-1}$, $\zeta>\frac{1}{\gamma}$ $\quad\forall,u,a_n,a_{n+1},\in\mathcal{K}$

Then the sequence $\{a_n\}$ is strong convergence to the solution $a\in\mathcal{K}$ of the Yosida variational inclusion problem (5)

Proof: we have,

$p(a_{n+1})\oplus p(a_n)\ \ge 0$ $[\because a\oplus b\ge 0, If\ a\propto b]$

$$\Rightarrow \mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,a_n)}[p(a_n)+\zeta\mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a_n)}(a_n)]\oplus\mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,a_{n-1})}[p(a_{n-1})+\zeta\mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a_{n-1})}(a_{n-1})]\ \ge 0$$

$$\Rightarrow \mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,a_n)}[p(a_n)+\zeta\mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a_n)}(a_n)]\oplus\ \mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,a_n)}[p(a_{n-1})+\zeta\mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a_{n-1})}(a_{n-1})]$$

$$\oplus\mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,a_n)}[p(a_{n-1})+\zeta\mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a_{n-1})}(a_{n-1})]\oplus\mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,a_{n-1})}[p(a_{n-1})+\zeta\mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a_{n-1})}(a_{n-1})]\ge 0 \tag{10}$$

Now we get from (iv) of proposition 2.3

$$\|p(a_{n+1})\oplus p(a_n)\|\le$$
$$\left\|\mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,a_n)}[p(a_n)+\zeta\mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a_n)}(a_n)]\oplus\ \mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,a_n)}[p(a_{n-1})+\zeta\mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a_{n-1})}(a_{n-1})]\right\|+$$
$$\left\|\mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,a_n)}[p(a_{n-1})+\zeta\mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a_{n-1})}(a_{n-1})]\oplus\mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,a_{n-1})}[p(a_{n-1})+\zeta\mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a_{n-1})}(a_{n-1})]\right\| \tag{11}$$

Again, utilize (3), (8), and (11), we get,

$$\Rightarrow\|p(a_{n+1})\oplus p(a_n)\|\le\theta\left\|[p(a_n)+\zeta\mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a_n)}(a_n)]\oplus\ [p(a_{n-1})+\zeta\mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a_{n-1})}(a_{n-1})]\right\|$$
$$+\mu\|a_n\oplus a_{n-1}\|\ \le\theta\left\|p(a_n)\oplus p(a_{n-1}))\right\|+\theta\zeta\left\|\mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a_n)}(a_n)\oplus\ \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a_{n-1})}(a_{n-1})]\right\|$$
$$+\mu\|a_n\oplus a_{n-1}\| \tag{12}$$

We have from the Lipschitz continuity of the Yosida variational inclusion problem and (i) of Proposition 2.3

$$\left\|\mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a_n)}(a_n)\oplus\ \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a_{n-1})}(a_{n-1})]\right\|=$$
$$\left\|\mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a_n)}(a_n)\oplus\mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a_{n-1})}(a_n)\oplus\mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a_{n-1})}(a_n)\oplus\mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a_{n-1})}(a_{n-1})]\right\|$$
$$\le\left\|\mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a_n)}(a_n)\oplus\mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a_{n-1})}(a_n)\right\|+\left\|\mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a_{n-1})}(a_n)\oplus\mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a_{n-1})}(a_{n-1})\right\|$$

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$\leq \mu' \| a_n \oplus a_{n-1} \| + \theta' \| a_n \oplus a_{n-1} \|$

$\leq (\mu' + \theta') \| a_n \oplus a_{n-1} \|$ (13)

Combining (12) and (13), we get

$\| p(a_{n+1}) \oplus p(a_n) \| \leq \theta \left\| p(a_n) \oplus p(a_{n-1}) \right\| + \mu \| a_n \oplus a_{n-1} \| + \theta\zeta(\mu' + \theta') \| a_n \oplus a_{n-1} \|$

Since p is strong convergence, then we get

$\| p(a_{n+1}) - p(a_n) \| \leq \theta\zeta_p \| a_n - a_{n-1} \| + \mu \| a_n - a_{n-1} \| + \theta\zeta(\mu' + \theta') \| a_n - a_{n-1} \|$

$\Rightarrow \| p(a_{n+1}) - p(a_n) \| \leq \{ \theta\zeta_p + \mu + \theta\zeta(\mu' + \theta') \} \| a_n - a_{n-1} \|$ (14)

Again, since p is strongly monotone, we have

$\| p(a_{n+1}) - p(a_n) \| \geq \delta_p \| a_{n+1} - a_n \|$

$\Rightarrow \frac{1}{\delta_p} \| p(a_{n+1}) - p(a_n) \| \geq \| a_{n+1} - a_n \|$ (15)

Joining (14) and (15), we get

$\| a_{n+1} - a_n \| \leq \frac{1}{\delta_p} \{ \theta\zeta_p + \mu + \theta\zeta(\mu' + \theta') \} \| a_n - a_{n-1} \|$

$\Rightarrow \| a_{n+1} - a_n \| \leq P(\theta) \| a_n - a_{n-1} \|$ (16)

Where, $P(\theta) = \frac{1}{\delta_p} \{ \theta\zeta_p + \mu + \theta\zeta(\mu' + \theta') \}$, From condition (A), we have $P(\theta) \leq 1$ and consequently, from (16), it follows that $\{ a_n \}$ is a Cauchy sequence in $\mathcal{K}$.

Since $\mathcal{K}$ is complete, we may assume that. $a_n \to a \in \mathcal{K}, n \to \infty$. This completes the proof.

YOSIDA RESOLVENT EQUATION PROBLEM

Now, we consider the subsequent resolvent equation problem, including XOR-operation.

Find $a, s \in \mathcal{K}$ such that

$\mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}(a) \oplus \zeta^{-1} \mathrm{J}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}(s) = 0$ (17)

Where $s = p(a) + \zeta \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}(x)$, $\mathrm{J}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})} = [\tau - \mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}]$

τ is the identity mapping, and $\zeta > 0$ is a constant.

Proposition 4.1. The element $a \in H$ is a solution to the Yosida variational inclusion problem including XOR-operation (5) if and only if $a, s \in H$ be a solution of the Yosida resolvent equation problem including XOR-operation (17). Provided $\mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}(a) \propto \mathrm{J}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}(s)$

Proof: Let $a \in H$ be a solution to the Yosida variational inclusion problem including XOR-operation (5). Then, by Lemma 3.1, it satisfies this equation:

$p(a) = \mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})} [p(a) + \zeta \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}(a)]$

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Since $s = p(a) + \zeta \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a)}(a)$

Then we get, $p(a) = \mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,a)}(s)$

Now we have, $s = \mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,a)}(s) + \zeta \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a)}(a)$

$\Rightarrow s - \mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,a)}(s) = \zeta \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a)}(a)$

$\Rightarrow [\, \tau - \mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,a)}](s) = \zeta \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a)}(a)$

$\Rightarrow \mathrm{J}_{\tau,\zeta}^{\mathfrak{D}(.,a)}(s) = \zeta \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a)}(a)$, where, $\mathrm{J}_{\tau,\zeta}^{\mathfrak{D}(.,a)} = [\, \tau - \mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,a)}]$

$\Rightarrow \zeta^{-1} \mathrm{J}_{\tau,\zeta}^{\mathfrak{D}(.,a)}(s) = \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a)}(a)$

$\Rightarrow \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a)}(a) \oplus \zeta^{-1} \mathrm{J}_{\tau,\zeta}^{\mathfrak{D}(.,a)}(s) = \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a)}(a) \oplus \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a)}(a) \quad [\because a \oplus a = 0]$

Thus, we have $\mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a)}(a) \oplus \zeta^{-1} \mathrm{J}_{\tau,\zeta}^{\mathfrak{D}(.,a)}(s) = 0$, which is the required Yosida resolvent equation problem, including XOR operation (17).

Conversely, let $a, s \in \mathcal{K}$ be the solution to the Yosida resolvent equation problem, including XOR-operation (17).

That is, $\mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a)}(a) \oplus \zeta^{-1} \mathrm{J}_{\tau,\zeta}^{\mathfrak{D}(.,a)}(s) = 0$

$\Rightarrow \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a)}(a) \oplus \zeta^{-1} \mathrm{J}_{\tau,\zeta}^{\mathfrak{D}(.,a)}(s) = \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a)}(a) \oplus \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a)}(a) \qquad [\because a \oplus a = 0]$

$\Rightarrow \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a)}(a) \oplus \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a)}(a) \oplus \zeta^{-1} \mathrm{J}_{\tau,\zeta}^{\mathfrak{D}(.,a)}(s) = \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a)}(a) \oplus \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a)}(a) \oplus \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a)}(a) \qquad [\because a \oplus a = 0]$

$\Rightarrow \zeta^{-1} \mathrm{J}_{\tau,\zeta}^{\mathfrak{D}(.,a)}(s) = \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a)}(a)$

$\Rightarrow \mathrm{J}_{\tau,\zeta}^{\mathfrak{D}(.,a)}(s) = \zeta \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a)}(a)$

$\Rightarrow \zeta \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a)}(a) = [\tau - \mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,a)}](s)$

$\Rightarrow \zeta \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a)}(a) = (s) - \mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,a)}(s)$

$\Rightarrow \zeta \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a)}(a) = p(a) + \zeta \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a)}(x) - \mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,a)}[p(a) + \zeta \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a)}(a)]$

$\Rightarrow p(a) = \mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,a)}[p(a) + \zeta \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a)}(a)]$

It may be inferred from Lemma 3.1 that $a \in \mathcal{K}$ is a solution to the Yosida variational inclusion problem including XOR operations (5). We build the following methods for addressing resolvent equation issues, such as the XOR-operations (17), based on assertion (17).

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Algorithm 4.2. Use the following approaches to compute the sequences $\{a_n\}$ and $\{s_n\}$ for every$a_0,\ s_0 \in \mathcal{K}$, we have

$$p(a_n) = \mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,a_n)}(s_n) \tag{18}$$

$$s_{n+1} = p(a_n) + \zeta \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a_n)}(a_n)\ , \tag{19}$$

τ is identity mapping and $\zeta > 0$ is a constant and $\mathrm{n} = 0,1,2, \dots \dots \dots$

By altering the Yosida resolvent equation problem, including XOR-operation (17). We propose a few techniques for solving resolvent equation problems, including XOR-operation (17).

That is, $s = \mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}(s) + \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}(x) \oplus \zeta^{-1}\mathrm{J}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}(s) + \mathrm{J}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}(s)$ (20)

Verification:

Now we have, $s - \mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}(s) = \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}(a) \oplus \zeta^{-1}\mathrm{J}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}(s) + \mathrm{J}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}(s)$

$$\Rightarrow [\tau - \mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}](s) = \zeta \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}(a) \oplus \zeta^{-1}\mathrm{J}_{\tau,\zeta}^{\square(.,\mathrm{a})}(s) + \mathrm{J}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}(s)$$

$$\Rightarrow \mathrm{J}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}(s) \oplus \mathrm{J}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}(s) = \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}(a) \oplus \zeta^{-1}\mathrm{J}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}(s) + \mathrm{J}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}(s) \oplus \mathrm{J}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}(s)$$

$$\Rightarrow \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}(x) \oplus \zeta^{-1}\mathrm{J}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}(s) = 0 \qquad [\because a \oplus a = 0]$$

Thus, $\mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}(a) \oplus \zeta^{-1}\mathrm{J}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}(s) = 0$

Utilizing (20), we build the techniques for solving Yosida resolvent equation problems including XOR-operation (17).

Algorithm 4.3. Use the following approaches to compute the sequences $\{a_n\}$ and $\{s_n\}$ for every $a_0,\ s_0 \in \mathcal{K}$, we have

$$p(a_n) = \mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,a_n)}(s_n) \tag{21}$$

$$s_{n+1} = \mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,a_n)}(s_n) + \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a_n)}(a_n) + \mathrm{J}_{\tau,\zeta}^{\mathfrak{D}(.,a_n)}(s_n) \oplus \zeta^{-1}\mathrm{J}_{\tau,\zeta}^{\mathfrak{D}(.,a_n)}(s_n) \tag{22}$$

Where τ is the identity mapping, $\zeta > 0$ is a constant and $\mathrm{n} = 0,1,2, \dots \dots \dots$

The resolvent equation problems can be rewritten for the positive step size δ using XOR-operation (17) as another form. $p(a) = p(a) \oplus \delta\,[\left(s - \mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}(s)\right) - \zeta \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}(a)]$ (23)

Verification: Now we have,

$$p(a) \oplus p(a) = p(a) \oplus p(a) \oplus \delta\,[\left(s - \mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}(s)\right) - \zeta \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}(a)]\ [\because a \oplus a = 0]$$

$$\Rightarrow 0 = \delta[\left(s - \mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}(s)\right) - \zeta \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}(a)]$$

$$\Rightarrow \delta\left(s - \mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}(s)\right) = \delta\zeta \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}(a)$$

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$\Rightarrow \left(s - R_{\tau,\zeta}^{\mathfrak{D}(.,a)}(s)\right) = \zeta Y_{\tau,\zeta}^{\mathfrak{D}(.,a)}(a)$

$\Rightarrow J_{\tau,\zeta}^{\mathfrak{D}(.,a)}(s) = \zeta Y_{\tau,\zeta}^{\mathfrak{D}(.,a)}(a)$

$\Rightarrow \zeta^{-1} J_{\tau,\zeta}^{\mathfrak{D}(.,a)}(s) = Y_{\tau,\zeta}^{\mathfrak{D}(.,a)}(a)$

$\Rightarrow Y_{\tau,\zeta}^{\mathfrak{D}(.,a)}(a) \oplus \zeta^{-1} J_{\tau,\zeta}^{\mathfrak{D}(.,a)}(s) = Y_{\tau,\zeta}^{\mathfrak{D}(.,a)}(a) \oplus Y_{\tau,\zeta}^{\mathfrak{D}(.,a)}(a)]$ $\quad [\because a \oplus a = 0]$

Thus $Y_{\tau,\zeta}^{\mathfrak{D}(.,a)}(a) \oplus \zeta^{-1} J_{\tau,\zeta}^{\mathfrak{D}(.,a)}(s) = 0$

Utilizing (23), we build the following techniques for solving Yosida resolvent equation problems, including XOR-operation (17) below.

Algorithm 4.4. Use the following approaches to compute the sequences $\{a_n\}$ and $\{s_n\}$ for every $a_0,\ s_0 \in \mathcal{K}$, we have

$$p(a_{n+1}) = p(a) \oplus \delta[(s_n - R_{\tau,\zeta}^{\mathfrak{D}(.,a_n)}(s_n)) - \zeta Y_{\tau,\zeta}^{\mathfrak{D}(.,a_n)}(a_n)] \tag{24}$$

Where $\zeta, \delta > 0$ are constant, τ is the identity mapping and $n = 0,1,2, \dots \dots \dots$

Theorem 4.5. Suppose all the mappings and axioms of theorem 3.3 remain the same, then the Yosida resolvent equation problem including XOR-operation (17) has a solution $x, s \in \mathcal{K}$ and the repetitional sequence $\{a_n\}$ and $\{s_n\}$represented by algorithm (4.2) strongly converges to a and s, respectively. Provided $a_{n+1} \propto a_n$, $s_{n+1} \propto s_n$ and $p(a_n) \propto p(a_{n-1})$, where $n = 0,1,2, \dots \dots \dots$

Proof: we have from proposition (4.1),

$$\|s_{n+1} \oplus s_n\| = \left\| [p(a_n) + \zeta Y_{\tau,\zeta}^{\mathfrak{D}(.,a_n)}(a_n)] \oplus [p(a_{n-1}) + \zeta Y_{\tau,\zeta}^{\mathfrak{D}(.,a_{n-1})}(a_{n-1})] \right\|$$

$$\Rightarrow \|s_{n+1} \oplus s_n\| \le \left\| [p(a_n) \oplus p(a_{n-1})] + \zeta [Y_{\tau,\zeta}^{\mathfrak{D}(.,a_n)}(a_n) \oplus Y_{\tau,\zeta}^{\mathfrak{D}(.,a_{n-1})}(a_{n-1})] \right\|$$

$$\Rightarrow \|s_{n+1} \oplus s_n\| \le \|p(a_n) \oplus p(a_{n-1})\| + \zeta \left\| Y_{\tau,\zeta}^{\mathfrak{D}(.,a_n)}(a_n) \oplus Y_{\tau,\zeta}^{\mathfrak{D}(.,a_{n-1})}(a_{n-1})] \right\| \tag{25}$$

$$\Rightarrow \|s_{n+1} - s_n\| \le \|p(a_n) - p(a_{n-1})\| + \zeta \left\| Y_{\tau,\zeta}^{\mathfrak{D}(.,a_n)}(a_n) \oplus Y_{\tau,\zeta}^{\mathfrak{D}(.,a_{n-1})}(a_{n-1})] \right\| \tag{26}$$

[$\because s_{n+1} \propto s_n$ and $p(a_n) \propto p(a_{n-1})$, for $n = 0,1,2, \dots \dots \dots$]

Now utilizing Lipschitz's continuity of p and (13), then equation (26) obtains to

$$\Rightarrow \|s_{n+1} - s_n\| \le \zeta_p \|a_n - a_{n-1}\| + \zeta(\mu' + \theta')\|a_n - a_{n-1}\|$$

[$\because p$ is strong convergence and $a_{n+1} \propto a_n$]

$$\Rightarrow \|s_{n+1} - s_n\| \le \{\zeta_p + \zeta(\mu' + \theta')\}\|a_n - a_{n-1}\| \tag{27}$$

We have, from (21) and (vi) of proposition 2.3

$$\|p(a_n) \oplus p(a_{n-1})\| = \left\| R_{\tau,\zeta}^{\mathfrak{D}(.,a_n)}(s_n) \oplus R_{\tau,\zeta}^{\mathfrak{D}(.,a_{n-1})}(s_{n-1}) \right\|$$

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$\Rightarrow \|p(a_n) \oplus p(a_{n-1})\| = \left\| \mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,a_n)}(s_n) \oplus \mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,a_n)}(s_{n-1}) \oplus \mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,a_n)}(s_{n-1}) \oplus \mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,a_{n-1})}(s_{n-1}) \right\|$

$\Rightarrow \|p(a_n) \oplus p(a_{n-1})\| \leq \left\| \mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,a_n)}(s_n) \oplus \mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,a_n)}(s_{n-1}) \right\|$

$+ \left\| \mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,a_n)}(s_{n-1}) \oplus \mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,a_{n-1})}(s_{n-1}) \right\|$ (28)

we get from Lemma (3), equations (8) and (28)

$\|p(a_n) \oplus p(a_{n-1})\| \leq \theta \|s_{n+1} \oplus s_n\| + \mu \|a_n \oplus a_{n-1}\|$ (29)

That is, $\|p(a_n) - p(a_{n-1})\| \leq \theta \|s_{n+1} - s_n\| + \mu \|a_n - a_{n-1}\|$ (30)

Since p is δ_p- strongly monotone, we have

$\|p(a_{n+1}) - p(a_n)\| \geq \delta_p \|a_{n+1} - a_n\|$

$\Rightarrow \frac{1}{\delta_p} \|p(a_{n+1}) - p(a_n)\| \geq \|a_{n+1} - a_n\|$ (31)

Combining (30) and (31)

$\|a_{n+1} - a_n\| \leq \frac{1}{\delta_p} \theta \|s_{n+1} - s_n\| + \frac{1}{\delta_p} \mu \|a_n - a_{n-1}\|$

$\Rightarrow \|a_{n+1} - a_n\| \leq \frac{\theta}{\delta_{p-\mu}} \|s_{n+1} - s_n\|$ (32)

Combining (32) and (27)

$\|s_{n+1} - s_n\| \leq \frac{\theta\zeta_p + \theta\zeta(\mu' + \theta')}{\delta_{p-\mu}} \|s_n - s_{n-1}\|$

$\Rightarrow \|s_{n+1} - s_n\| \leq s(\theta) \|s_n - s_{n-1}\|$ Where, $S(\theta) = \frac{\theta\zeta_p + \theta\zeta(\mu' + \theta')}{\delta_{p-\mu}}$

Using axioms (A), after that $S(\theta) < 1$ and so $\{s_n\}$ is a Cauchy sequence in $\mathcal{K}$. Thus, there exists $s \in \mathcal{K}$ such that. $s_n \to s$,$as\ n \to \infty$. Moreover, from (32), obviously $\{a_n\}$ is a Cauchy sequence in $\mathcal{K}$, then there exists $a \in \mathcal{K}$ such that $a_n \to a\ \ as\ n \to \infty$.

Using the continuity of operators $p, \mathfrak{D}$, and $\mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.\ ,\ .)}$, We have $s = p(a) + \zeta \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a)}(a)$ which is the same result of preposition 4.1.

CONVERGENCE EXPERIMENT

Experiment 1. Suppose $\mathcal{K} = \mathbb{R}$ involving inner product$\langle .,. \rangle\ and\ \|\ .\ \|$. Again, consider $p : \mathcal{K} \to \mathcal{K}$ be a single-valued mapping and $\mathfrak{D} : \mathcal{K} \times \mathcal{K} \to 2^{\mathcal{K}}$ be a multi-valued mapping such that, $\mathfrak{D}(a, b) = \{\frac{7}{12} a + b\}$

$p(a) = \frac{9}{7} a - 1, \forall a, b \in \mathcal{K}$

(i). Suppose $\mathfrak{D}$ is γ-ordered rectangular mapping, then there exist $v_a = \frac{19}{12} a \in \mathfrak{D}(\ .\ , a)$ and $v_b = \frac{19}{12} b \in \mathfrak{D}(\ .\ , b)$ we get

$$\langle (v_a \odot v_b) - (a \oplus b) \rangle = \langle (v_a \oplus v_b), (a \oplus b) \rangle$$

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$$= \langle \tfrac{19}{12}a \oplus \tfrac{19}{12}b, a \oplus b \rangle$$
$$= \frac{19}{12} \langle a \oplus b, a \oplus b \rangle$$
$$= \frac{19}{12} \|a \oplus b\|^2$$
$$\geq \frac{1}{5} \|a \oplus b\|^2$$

Thus, $\mathfrak{D}$ is $\gamma = \frac{1}{5}$ - ordered rectangular mapping.

a. Suppose p is ζ_p- Lipschitz continuous and δ_p- strongly monotone mapping.

We have $\|p(a) - p(b)\| = \left\| (\tfrac{9}{7}a - 1) - (\tfrac{9}{7}b - 1) \right\|$

$$= \tfrac{9}{7}\|a - b\|$$
$$\leq 3\|a - b\|$$

Hence, p is $\zeta_p = 3$ - Lipschitz continuous mapping.

And $\langle p(a) - p(\square), a - b \rangle = \langle (\tfrac{9}{7}a - 1) - (\tfrac{9}{7}b - 1) \rangle$

$$= \tfrac{9}{7}\langle a - b, a - b \rangle$$
$$\geq \frac{1}{7} \|a - b\|^2$$

Thus, p is $\delta_p = \frac{1}{7}$ - strongly monotone mapping.

b. Consider $\zeta = 6$, then evaluate the resolvent operator as

$$\mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}(a) = [\tau + \zeta \mathfrak{D}(.,a)]^{-1}(a)$$
$$= [a + \zeta \mathfrak{D}(a,a)]^{-1}$$
$$= [a + 6 \times \tfrac{19}{12}a]^{-1}$$
$$= [\tfrac{21}{2}a]^{-1}$$
$$= \tfrac{2}{21}a$$

$and \left\| \mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}(a) \oplus \mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}(b) \right\| = \left\| \tfrac{2}{21}a \oplus \tfrac{2}{21}b \right\|$

$$= \frac{2}{21} \|a \oplus b\|$$
$$\leq 5\|a \oplus b\|$$

That is, $\theta = \frac{1}{\gamma\zeta - 1} = 5$, $where, \zeta = 6, \gamma = \frac{1}{5}$ then $\mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}$ is Lipschitz continuous.

c. Again, we have from the Yosida approximation operator.

$\mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,\mathrm{a})}(a) = \frac{1}{\zeta}[\tau - R(.,a)](a)$

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$= \frac{1}{\zeta}[a - R(a,a)]$

$= \frac{1}{6}[a - \frac{2}{21}a]$

$= \frac{1}{6} \times \frac{19}{21}a$

$= \frac{19}{126}a$

And $\left\| \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a)}(a) \oplus \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a)}(b) \right\| = \left\| \frac{19}{126}a \oplus \frac{19}{126}b \right\|$

$= \frac{19}{126}\|a \oplus b\|$

$\leq 30\|a \oplus b\|$

That is, $\mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a)}(a)$ is Lipschitz continuous with constant$\theta' = \frac{\zeta}{\gamma\zeta - 1} = 30$, where$\zeta = 6$, $\gamma = \frac{1}{5}$

d. We get from technique (7).

$p(a_{n+1}) = \mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,a_n)}[p(a_n) + \zeta \mathrm{Y}_{\tau,\zeta}^{\mathfrak{D}(.,a_n)}(a_n)]$

$\Rightarrow \frac{9}{7}a_{n+1} - 1 = \mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,a_n)}[\frac{9}{7}a_n - 1 + 6 \times \frac{19}{126}a_n]$

$\Rightarrow a_{n+1} = 0.16222a_n + 0.70377$

Now, using MATLAB-R2024a, we get an excellent graph for various initial values $a_0 = -3, -1, 1, 2.5, 4$, the sequence $\{a_n\}$ is swiftly converged at $a^* = 0.83999$(up to five decimals) after six iterations. The diagram and estimate chart are given below.

Estimate Chart:

No. of iterations	a_0= -3.0 a_n	a_0= -1.0 a_n	a_0= 1.0 a_n	a_0= 2.5 a_n	a_0= 4.0 a_n
1	0.21710	0.54150	0.86590	1.10920	1.35250
2	0.73891	0.79153	0.84415	0.88361	0.92308
3	0.82355	0.83209	0.84062	0.84702	0.85342
4	0.83728	0.83866	0.84005	0.84109	0.84213
5	0.83951	0.83973	0.83996	0.84012	0.84029
6	0.83987	0.83990	0.83994	0.83997	0.84000
7	0.83993	0.83993	0.83994	0.83994	0.83995
8	0.83994	0.83994	0.83994	0.83994	0.83994
9	0.83994	0.83994	0.83994	0.83994	0.83994
13	0.83994	0.83994	0.83994	0.83994	0.83994
15	0.83994	0.83994	0.83994	0.83994	0.83994
17	0.83994	0.83994	0.83994	0.83994	0.83994
19	0.83994	0.83994	0.83994	0.83994	0.83994

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Graphical Representation:

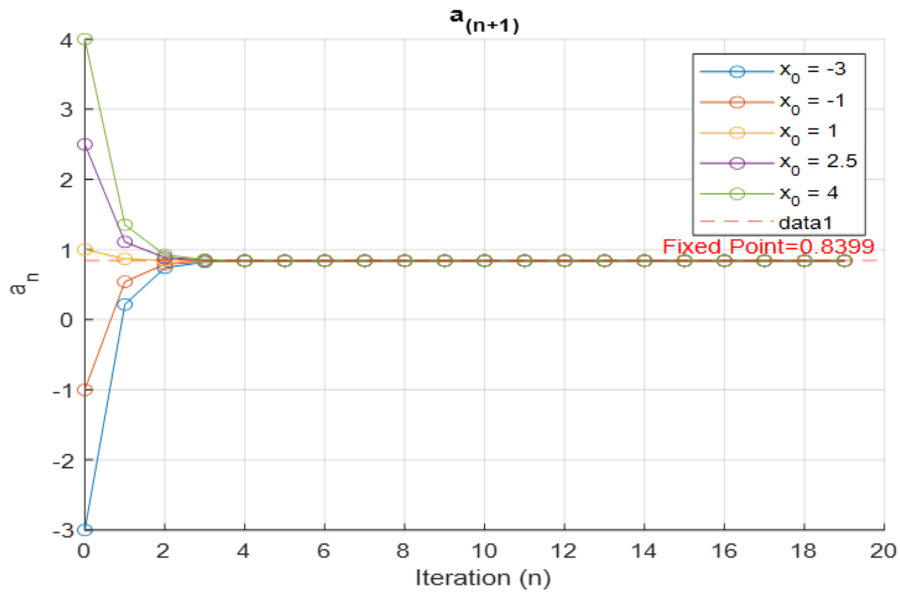

Diagram: 1

Experiment 2. Consider $\mathcal{K} = \mathbb{R}$ involving the inner product$\langle .,. \rangle$ *and* $\| \, . \, \|$. Again, consider $p : \mathcal{K} \to \mathcal{K}$ be a single-valued mapping and $\mathfrak{D} : \mathcal{K} \times \mathcal{K} \to 2^{\mathcal{K}}$ be a multi-valued mapping such that $\mathfrak{D}(a,b) = \{ \frac{5}{9} a + b \}$ and $p(a) = \frac{7}{5} a - 1, \forall a, b \in \mathcal{K}$

(a) Consider $\mathfrak{D}$ is γ-ordered rectangular mapping, then there exist $v_a = \frac{14}{9} a \in \mathfrak{D}(\,.\,, a)$ and $v_b = \frac{14}{9} b \in \mathfrak{D}(\,.\,, b)$ we get

$$\langle (v_a \odot v_b) - (a \oplus b) \rangle = \langle (v_a \oplus v_b), (a \oplus b) \rangle$$

$$= \langle \frac{14}{9} a \oplus \frac{14}{9} b, a \oplus b \rangle$$

$$= \frac{14}{9} \langle a \oplus b, a \oplus b \rangle$$

$$= \frac{14}{9} \| a \oplus b \|^2$$

$$\geq \frac{1}{2} \| a \oplus b \|^2$$

Thus, $\mathfrak{D}$ is $\gamma = \frac{1}{2}$- ordered rectangular mapping.

a. Suppose p is ζ_p- Lipschitz continuous and δ_p- strongly monotone mapping.

We have $\| p(a) - p(b) \| = \left\| (\frac{7}{5} a - 1) - (\frac{7}{5} b - 1) \right\|$

$$= \frac{7}{5} \| a - b \|$$

$$\leq 2 \, \| a - b \|$$

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Hence, p is $\zeta_p = 3$ - Lipschitz continuous mapping.

And $\langle p(a) - p(b), a - b \rangle = \langle (\frac{7}{5} a - 1) - (\frac{7}{5} b - 1) \rangle$

$$= \frac{7}{5} \langle a - b, a - b \rangle$$

$$\geq \frac{1}{5} \|a - b\|^2$$

Thus, p is $\delta_p = \frac{1}{5}$ - strongly monotone mapping

b. Consider $\zeta = 3$, then evaluate the resolvent operator as

$$R_{\tau,\zeta}^{\mathfrak{D}(.\ ,a)}(a) = [\tau + \zeta\mathfrak{D}(.,a)]^{-1}(a)$$

$$= [a + \zeta\mathfrak{D}(a,a)]^{-1}$$

$$= [a + 3 \times \frac{14}{9} a]^{-1}$$

$$= [\frac{17}{3} a]^{-1}$$

$$= \frac{3}{17} a$$

and $\left\| R_{\tau,\zeta}^{\mathfrak{D}(.,a)}(a) \oplus R_{\tau,\zeta}^{\mathfrak{D}(.,a)}(b) \right\| = \left\| \frac{3}{17} a \oplus \frac{3}{17} b \right\|$

$$= \frac{3}{17} \|a \oplus b\|$$

$$\leq 2 \|a \oplus b\|$$

That is, $\theta = \frac{1}{\gamma\zeta - 1} = 2$, *where*, $\zeta = 3, \gamma = \frac{1}{2}$, thus $R_{\tau,\zeta}^{\mathfrak{D}(.,a)}$ is Lipschitz continuous.

c. Again, we have from the Yosida approximation operator.

$$Y_{\tau,\zeta}^{\mathfrak{D}(.,a)}(a) = \frac{1}{\zeta} [\tau - \square(.,a)](a)$$

$$= \frac{1}{\zeta} [\tau - R(a\ ,a)]$$

$$= \frac{1}{3} [a - \frac{3}{17} a]$$

$$= \frac{1}{3} \times \frac{14}{17} a$$

$$= \frac{14}{51} a$$

And $\left\| Y_{\tau,\zeta}^{\mathfrak{D}(.,a)}(a) \oplus Y_{\tau,\zeta}^{\mathfrak{D}(.,a)}(b) \right\| = \left\| \frac{14}{51} a \oplus \frac{14}{51} b \right\|$

$$= \frac{14}{51} \|a \oplus b\|$$

$$\leq 6 \|a \oplus b\|$$

That is, $Y_{\tau,\zeta}^{\mathfrak{D}(.,a)}(a)$ is Lipschitz continuous with constant $\theta' = \frac{\zeta}{\gamma\zeta - 1} = 6$ where, $\zeta = 3, \gamma = \frac{1}{2}$

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d. We get from technique (8)

$$p(a_{n+1}) = \mathrm{R}_{\tau,\zeta}^{\mathfrak{D}(.,a_n)}[p(a_n) + \zeta \Upsilon_{\tau,\zeta}^{\mathfrak{D}(.,a_n)}(a_n)]$$

$$\Rightarrow \tfrac{7}{5}a_{n+1} - 1 = \tfrac{3}{17}[\tfrac{7}{5}a_n - 1 + 3 \times \tfrac{14}{51}a_n]$$

$$\Rightarrow \ a_{n+1} = 0.28022a_n + 0.58822$$

Again, using MATLAB-R2024a, we get an excellent graph for various initial values $a_0 = -4, -2,\ 3,\ 6,$ the sequence $\{a_n\}$ is swiftly converged at $a^* = 0.81717$ (up to five decimals) after nine iterations. The diagram and estimate chart are given below.

Estimate Chart:

No. of iterations	a_0= -3.0 a_n	a_0= -1.0 a_n	a_0= 1.0 a_n	a_0= 2.5 a_n
1	-0.53260	0.02780	1.42881	2.26943
2	0.43897	0.59599	0.98855	1.22412
3	0.71120	0.75520	0.86519	0.93119
4	0.78748	0.79981	0.83063	0.84912
5	0.80885	0.81231	0.82094	0.82612
6	0.81652	0.81581	0.81823	0.81968
7	0.81652	0.81679	0.81747	0.81968
8	0.81699	0.81706	0.81725	0.81737
9	0.81712	0.81714	0.81719	0.81723
11	0.81717	0.81717	0.81717	0.81717
13	0.81717	0.81717	0.81717	0.81717
15	0.81717	0.81717	0.81717	0.81717
17	0.81717	0.81717	0.81717	0.81717
19	0.81717	0.81717	0.81717	0.81717

Graphical Representation:

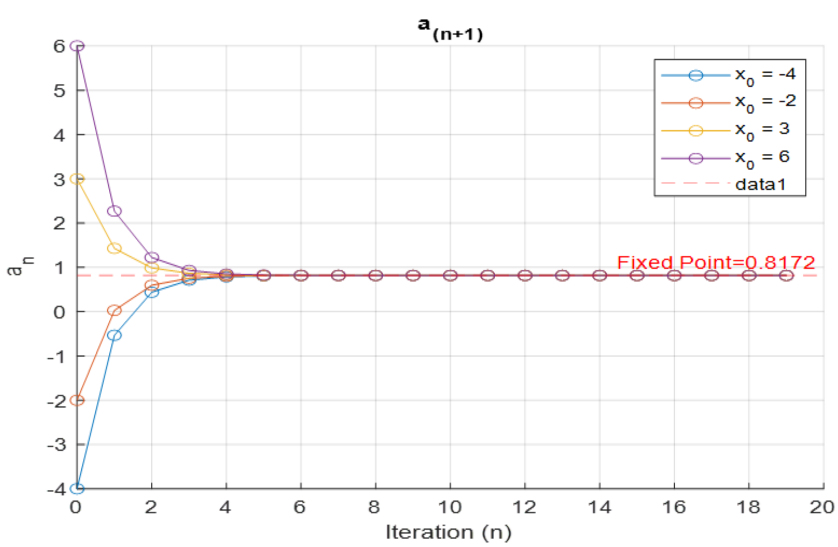

Diagram: 2

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CONCLUSION

A problem of inclusion is presented in a real ordered Hilbert space using the Yosida approximation operator, a multi-valued mapping, a single-valued mapping, and an XOR-operator. The study also includes a problem with an equipotential resolvent equation with the XOR operators. Certain algorithms are specifically designed to address the Yosida inclusion problem and solve equations that involve the XOR-operator. Existence and convergence outcomes are demonstrated for each issue. Both mathematical models exhibit rapid convergence, which contributes to the attainment of our optimal solution.

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