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THE GENERALIZED ALGORITHM FOR FRACTIONAL CONTINUITY OF POLYNOMIAL SPLINES

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Abstract

Spline is a collection of mathematical functions is used to draw smooth curves through a set of points. Splines have applications in computer graphics, image processing, robotics, planning, and data interpolation. Fractional continuity is a mathematical and computer-graphics term that enhances the concept of continuity in curve and surface modeling. In this paper, the concept of fractional order continuity of generalized spline functions, an interpolating continuity class $C^{n-1+\alpha}, 0 < \alpha < 1,$, is presented, which gives visually piece wise smooth curves. Firstly the general algorithm for generalized polynomial splines is presented, after that it elaborate with different degree of splines. The special cases of the proposed work are also presented. The Caputo left hand and right hand fractional derivative are used in proposed algorithm. The curve produced by proposed algorithm are also control with the help of shape parameters *u* and *v*.

Keywords: Fractional Continuity; Splines; Caputo Derivative; Fractional Calculus.

AMS Subject Classifications: 26A33, 65D07, 26A15, 26B05.

1. INTRODUCTION

Splines are piecewise polynomials of degree *n* that satisfy at their joints in terms of both function values and first $n-1$ derivatives. Piecewise spline-based techniques are the common methods adopted in computer-aided geometric design. For interpolating or approximating univariate data, spline functions with particular geometric shapes or properties like nonnegativity, convexity, or monotonicity are often used.

Designers find it easy to implement these strategies due to the spline algorithm's truthfulness; in fact, a lot of effort has been done in this field, and researchers are still working on a range of ways by improving them to make them more diversified. Spline interpolation aims to produce an interpolation formula that is smooth and continuous at the interpolating locations as well as inside the intervals.

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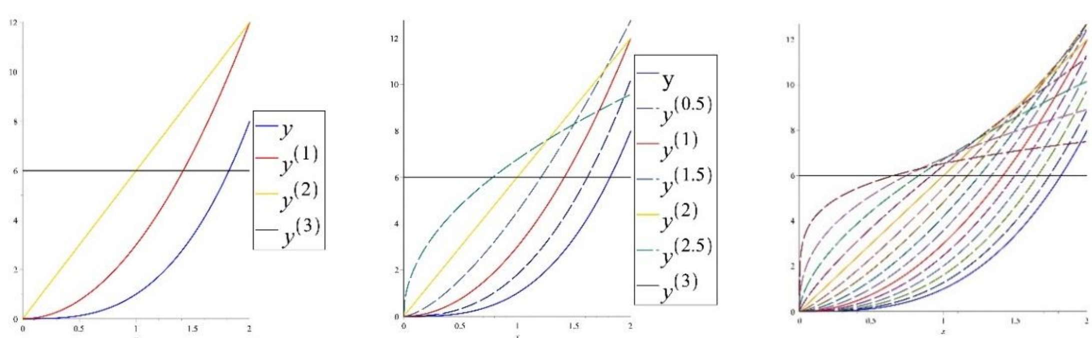
Recently, excellent work has been done on the piece wise polynomial spline [4]. Spline as a useful and convenient statistical tool [2], splines functions in Euclidean space R^d [4], first and second order geometric continuity of system of interpolating system of splines functions [5], a subclass of Catmull-Rom splines that has shape parameters [5]. There is a description of spline techniques for radial equation solution for continuum state [8].

Fractional calculus is a generalization of classical calculus that focusses on noninteger order integration and differentiation. Since its debut in 1695, Ludwig Hoffmann has been asking Gottfried Wilhelm Leibniz what would happen if we were to employ $k = \frac{1}{2}$ in $\frac{d^k}{dx^k}$.

The fractional derivative of the trigonometric function was first suggested by mathematician Joseph Fourier in 1822. Liouville (1832) made the first important study into fractional calculus, applying the Fourier fractional integral and the Abel solution based on potential theory. Liouville established the fractional derivative of exponential functions and on certain restricted functions, he naturally extended his discovery of the ordinary derivative to the fractional derivative.

Reisz (1949) introduced a theory of fractional integration for functions with multiple variables. However, Reimann and Liouville's definition of fractional derivative is ambiguous, as it doesn't equal zero for constant functions. Caputo (1967) simplified this definition using series expansion, revealing that the derivative is not an exponential function but a more comprehensive Mittag-Leffler function. This disagreement in fractional calculus has led to different results for the same function.

A fundamental feature of fractional calculus that is applicable to fractional continuity of spline functions is that there are no rapid changes, discontinuities, or jumps visible on the graphs of a function or its derivatives. As an example, Fig.1 displays the graphs of x^3, its first, second, and third order derivatives as well as the fractional order derivative between the integer order.



(a) (b) (c)

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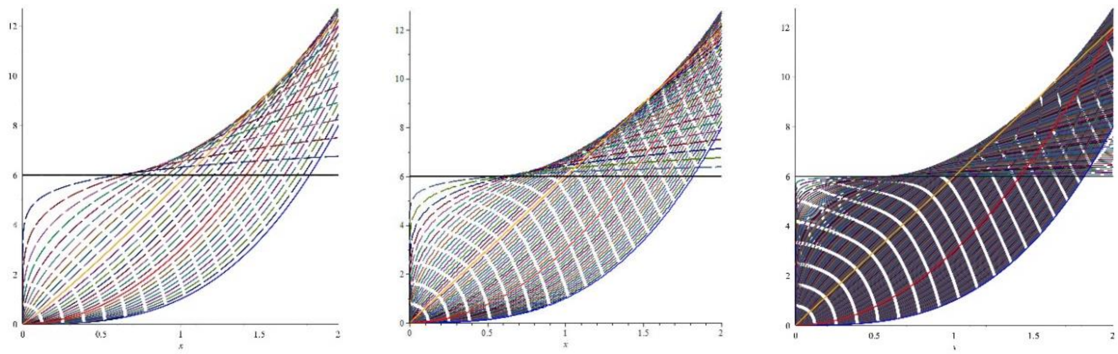

(d) (e) (f)

Figure 1: Solid lines are the graphs of function and its integral order derivatives and the dash-lines are the graphs of the fractional order derivatives between the integral order derivatives

(a) Represent graphs of $y = x^3$ and its derivatives. (b) Represent graphs integer order derivatives and one fractional order derivative between integral order. (c) Represent graphs integer order derivatives and five fractional order derivative between integral order. (d) Represent graphs integer order derivatives and ten fractional order derivative between integral order. (e) Represent graphs integer order derivatives and twenty fractional order derivative between integral order. (f) Represent graphs integer order derivatives and hundred fractional order derivative between integral order.

1.1 Motivation

The fractional continuity of polynomial spline functions are discussed in [17] and [18]. The cubic spline posses the fraction continuity $\alpha \in (0,1)$ and biquadratic spline of order $\alpha \in (1,2)$. The question is that, is it possible to increase the order of continuity of these spline up to the degree of polynomials? Are there other polynomial spline that have the fractional continuity?

In this paper general algorithm is developed to find the fractional continuity of spline of any order $<$ minimum degree of polynomials. By applying the algorithm, the fractional continuity of cubic can be determine of order $\alpha \in (2,3)$ and Biquadratic spline can have the continuity of fractional order $\alpha \in (3,4)$. Also new splines of polynomial functions can be generated that have the fractional continuity of order $\leq$ degree of spline defined in algorithm. Polynomial in the existing spline are of same degree. But the polynomials in this article are of same and different degrees.

The paper is unfolded as follows. Section 2 is for the construction of general algorithm to find the fractional continuity of any order of the spline of polynomial of any degree. Section 3 is devoted for application and analysis of the general algorithm for different degree of polynomials and level of continuity. This section have two cases, in case-I the polynomials are of same degree, in case-II, the degree of the polynomials are of different degrees. Section 4 draw the conclusions of the whole work done in this article.

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2. CONSTRUCTION OF GENERAL ALGORITHM

In this section, a general algorithm is developed to find the fractional continuity of spline of polynomial functions. Consider a general polynomial spline function

$$S(t) = \begin{cases} P_i(t) & \text{if} \quad t \in [t_i, t_{i+1}], \\ P_{i+1}(t) & \text{if} \quad t \in [t_{i+1}, t_{i+2}]. \end{cases} \tag{2.1}$$

Where, $P_i(t)$ and $P_{i+1}(t)$ are polynomials of degree p and q respectively defined as.

$$P_i(t) = \sum_{j=0}^{p} a_{i,j}(t - t_i)^j.$$
$$P_{i+1}(t) = \sum_{j=0}^{q} a_{i,j}(t - t_{i+1})^j. \tag{2.2}$$

Where $p, q \geq 2$, define $n = min\{p, q\}$. The $P_i(t)$ contained the points (t_i, y_i) and (t_{i+1}, y_{i+1}). The $P_{i+1}(t)$ contained the points (t_{i+1}, y_{i+1}) and (t_{i+2}, y_{i+2}). These are the initial conditions for which $P_i(t_i) = y_i$, $P_i(t_{i+1}) = y_{i+1}$, $P_{i+1}(t_{i+2}) = y_{i+2}$, shows that

$$a_{i,0} = y_i, \qquad a_{i+1,0} = y_{i+1}. \tag{2.3}$$

$$\sum_{j=0}^{q} a_{i+1,j} h_{i+1}^j = y_{i+2} - y_{i+1}. \tag{2.4}$$

Where, $h_{i+1} = t_{i+2} - t_{i+1}$.

To construct of splines that have fractional continuity of order $\alpha \in (\mu - 1, \mu)$ at the point (t_{i+1}, y_{i+1})., where $1 \leq \mu \leq n$. The real constants $a_{i,0}, a_{i,1}, \cdots, a_{i,p}$, $a_{i+1,0}, a_{i+1,1}, \cdots, a_{i+1,q}$ are required. These are $p + q + 2$ in numbers, $a_{i,0}$ and $a_{i+1,0}$ already given in (2.3), remaining $(p + q)$ constants can be determined using the following conditions

$$P_i^{(r)}(t_{i+1}) = P_{i+1}^{(r)}(t_{\square+1}), \qquad r = 0,1,2,\cdots,\mu. \tag{2.5}$$

and

$$P_i^{(\alpha)}(t_{i+1}) = -P_{i+1}^{(\alpha)}(t_{i+1}), \quad \alpha \in (\mu - 1, \mu). \tag{2.6}$$

The $P_i(t)$ and $P_{i+1}(t)$ are left hand and right hand Caputo derivative of order α at the $t = t_{i+1}$. The r^{th} order derivatives of (2.2) are given as,

$$P_i^{(r)}(t) = \sum_{R=r}^{p} {}^{R}P_r a_{i,R}(t - t_i)^{R-r},$$
$$P_{i+1}^{(r)}(t) = \sum_{R=r}^{q} {}^{R}P_r a_{i+1,R}(t - t_{i+1})^{R-r}. \tag{2.7}$$

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After putting (2.7) in (2.5), which gives the following results

$$\sum_{k=r}^{p} {}^{R}P_r a_{i,R}(t_{i+1}-t_i)^{R-r} = \sum_{R=r}^{q} {}^{R}P_r a_{i+1,R}(t_{i+1}-t_{i+1})^{R-r}.$$

This implies,

$$\sum_{k=r}^{p} {}^{R}P_r a_{i,R}(h_i)^{R-r} = r!\, a_{i+1,r}. \tag{2.8}$$

The fractional differentiability condition defined in (2.6)

$$\begin{aligned} P_i^{(\alpha)}(t_{i+1}) &= \frac{1}{\Gamma(\mu-\alpha)}\int_u^{t_{i+1}} \frac{P_{i,j}^{(\mu)}(\eta)}{(t_{i+1}-\eta)^{\alpha-\mu+1}}\,d\eta, \\ &= \frac{1}{\Gamma(\mu-\alpha)}\sum_{K=0}^{p-\mu} a_{i,\mu+K}\frac{(\mu+K)!}{K!}(-1)^K \sum_{\delta=0}^{K}\binom{K}{\delta}(-h_i)^{K-\delta}\frac{(t_{i+1}-u)^{\delta+\mu-\alpha}}{\delta+\mu-\alpha} \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} P_{i+1}^{(\alpha)}(t_{i+1}) &= \frac{1}{\Gamma(\mu-\alpha)}\int_{t_{i+1}}^{v} \frac{P_{i+1,j}^{(\mu)}(\eta)}{(\eta-t_{i+1})^{\alpha-\mu+1}}\,d\eta, \\ &= \frac{1}{\Gamma(\mu-\alpha)}\sum_{k=0}^{q-\mu} a_{i+1,\mu+k}\frac{(\mu+k)!}{k!}\frac{(v-t_{i+1})^{\mu+k-\alpha}}{\mu+k-\alpha}. \end{aligned}$$

From (2.6)

$$\sum_{K=0}^{p-\mu} a_{i,\mu+K}\frac{(\mu+K)!}{K!}(-1)^K A_{p,K,\mu} + \sum_{k=0}^{q-\mu} a_{i+1,\mu+k}\frac{(\mu+k)!}{k!}B_{q,k,\mu} = 0. \tag{2.10}$$

Where,

$$A_{p,K,\mu} = \sum_{\delta=0}^{K}\binom{K}{\delta}(-h_i)^{K-\delta}\frac{(t_{i+1}-u)^{\delta+\mu-\alpha_\theta}}{\delta+\mu-\alpha_\theta}, \quad B_{q,k,\mu} = \frac{(v-t_{i+1})^{\mu+k-\alpha_\theta}}{\mu+k-\alpha_\theta}.$$

$$K = 0,1,2,\cdots p-\mu,$$

$$k = 0,1,2,\cdots q-\mu,$$

$$\theta = 1,2,3,\cdots, p+q-\mu-2.$$

To determine these $p+q$ constants, $p+q$ equations are required. An equation is obtained from (2.4)

$\mu+1$ equations are obtained from (2.5) remaining $p+q-\mu-2$ number of equations are obtained from (2.10) by assigning the different values to $\alpha_\theta \in (\mu-1,\mu)$. There are three

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possible outcomes while solving the system of linear equations, unique solution, infinite solution and no solution. inconsistency problem can be solved by using the wide support size of the spline functions. when system have infinite many solution then there are two techniques one is wide the support size and second is to introduce free variables, spline also controlled by these free variables. To explain the working of the proposed algorithm, some examples are given for different values of p, q and μ.

3. APPLICATIONS AND ANALYSIS

In this section, the generalized algorithm is explained by assigning different values to p, q and μ. It include two sections in first section $p = q$ and in second section $p \neq q$.

3.1 Case-I: (p = q)

In this case, the polynomials spline functions have same degree like $3, 4$ and 5 and special cases of proposed work are discussed.

If $p = q = 3$ and $\mu = 1$, in proposed algorithm, which leads to [17].

If $p = q = 4$ and $\mu = 2$, in proposed algorithm, which leads to [18].

3.1.1 For $\boldsymbol{p = q = 3}$ and $\boldsymbol{\mu = 2}$

Furthermore for p = q = 3 and μ = 2, the spline function is given as

$$\begin{aligned} P_i(t) &= a_{i,0} + a_{i,1}(t - t_i) + a_{i,2}(t - t_i)^2 + a_{i,3}(t - t_i)^3, \\ P_{i+1}(t) &= a_{i+1,0} + a_{i+1,1}(t - t_i) + a_{i+1,2}(t - t_i)^2 + a_{i+1,3}(t - t_i)^3. \end{aligned} \tag{3.1}$$

From (2.4), we get

$$a_{i+1,1}h_{i+1} + a_{i+1,2}h_{i+1}^2 + a_{i+1,3}h_{i+1}^3 = y_{i+2} - y_{i+1}. \tag{3.2}$$

After putting $\mu = 2$ in (2.7), the following three equations are obtained

$$\begin{aligned} a_{i,0} + a_{i,1}h_i + a_{i,2}h_i^2 + a_{i,3}h_i^3 &= a_{i+1,0}, \\ a_{i,1} + 2a_{i,2}h_i + 3a_{i,3}h_i^2 &= a_{i+1,1}, \\ a_{i,2} + 3a_{i,3}h_i &= a_{i+1,2}. \end{aligned} \tag{3.3}$$

The (2.10) for $p = q = 3$ and $\mu = 2$ takes the form

$$2a_{i,2}A_{3,0,2} - 3a_{i,3}A_{3,1,2} + 2a_{i+1,2}B_{3,0,2} + 3a_{i+1,3}B_{3,1,2} = 0. \tag{3.4}$$

For $\theta = 1$ and $\theta = 2$, two linear equations are obtained, therefore form (3.2),(3.3) and (3.4), consists six equations. After solve these equations for the coefficients of the polynomials $P_i(t)$ and $P_{i+1}(t)$ to form a cubic spline that have the fractional continuity of fractional order $\alpha \in (1,2)$. The graphical representation is given in Fig. 2b.

3.1.2 For $\boldsymbol{p = q = 3}$ and $\boldsymbol{\mu = 3}$

If $p = q = 3, \mu = 3$, the set of equations (2.7) another equation $a_{i,3} = a_{i+1,3}$ will be added to the set of equations (2.7) and (2.10) modified as,

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$$a_{i,3}A_{3,0,3} + a_{i+1,3}B_{3,0,3} = 0. \tag{3.5}$$

where,

$$A_{3,0,3} = \sum_{\delta=0}^{0}\binom{0}{\delta}(-h_i)^{0-\delta}\frac{(t_{i+1}-u)^{\delta+3-\alpha_\theta}}{\delta+3-\alpha_\theta},$$
$$= \frac{(-h_i)(t_{i+1}-u)^{2-\aleph_\theta}}{2-\alpha_\theta} + \frac{(t_{i+1}-u)^{3-\aleph_\theta}}{3-\alpha_\theta},$$
$$B_{3,0,3} = \frac{(v-t_{i+1})^{3-\alpha_\theta}}{3-\alpha_\theta}, \quad \theta=1.$$

So the only one equation is needed to find all coefficients of the cubic spline, that can be obtained by setting the value of αθ ∈ (2,3). The graphical represents of this case is presented in Fig. 2c.

3.1.3 For p = q = 4 and μ = 1

For p = q = 4, μ = 1 the set of equations by applying continuity and differentiability conditions is given below.

$$P_i(t) = a_{i,0} + a_{i,1}(t-t_i) + a_{i,2}(t-t_i)^2 + a_{i,3}(t-t_i)^3 + a_{i,4}(t-t_i)^4,$$
$$P_{i+1}(t) = a_{i+1,0} + a_{i+1,1}(t-t_i) + a_{i+1,2}(t-t_i)^2 + a_{i+1,3}(t-t_i)^3 + a_{i+1,4}(t-t_i)^4. \tag{3.6}$$

The (2.4) takes the form

$$a_{i+1,1}h_{i+1} + a_{i+1,2}h_{i+1}^2 + a_{i+1,3}h_{i+1}^3 + a_{i+1,4}h_{i+1}^4 = y_{i+2} - y_{i+1}, \tag{3.7}$$

After putting $\mu = 1$ in (2.7), the following three equations are obtained

$$a_{i,0} + a_{i,1}h_i + a_{i,2}h_i^2 + a_{i,3}h_i^3 + a_{i,4}h_i^4 = a_{i+1,0}, \tag{3.8}$$
$$a_{i,1} + 2a_{i,2}h_i + 3a_{i,3}h_i^2 + 4a_{i,4}h_i^3 = a_{i+1,1},$$

The (2.10) takes the form

$$a_{i,1}A_{4,0,1} - 2a_{i,2}A_{4,1,1} + 3a_{i,3}A_{4,2,1} - 4a_{i,3}A_{4,3,1} + a_{i+1,1}B_{4,0,1} + 2a_{i+1,2}B_{4,1,1} + 3a_{i+1,3}B_{4,2,1} + 4a_{i+1,4}B_{4,3,1} = 0. \tag{3.9}$$

where,

$$A_{4,K,1} = \sum_{\delta=0}^{K}\binom{K}{\delta}(-h_i)^{K-\delta}\frac{(t_{i+1}-u)^{\delta+1-\alpha_\theta}}{\delta+1-\alpha_\theta}, B_{4,k,1} = \frac{(v-t_{i+1})^{1+k-\alpha_\theta}}{1+k-\alpha_\theta},$$
$$k = 0,1,2,3, \quad \theta = 1,2,\cdots,5.$$

After solving the above equation, one can get the spline with fractional continuity. The graphical representation is given in Fig. 3 (a).

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3.1.4 For $\boldsymbol{p = q = 4}$ and $\boldsymbol{\mu = 3}$

For $p = q = 4, \mu = 3$ the system of equations include two equations

$$a_{i,2} + 3a_{i,3}h_i + 6a_{i,4}h_i^2 = a_{i+1,2},$$
$$a_{i,3} + 4a_{i,4}h_i = a_{i+1,3}, \quad (3.10)$$

After putting $\mu = 3$ in (2.5) the following three equations are obtained

$$\sum_{K=0}^{1} a_{3+K} \frac{(\mu+K)!}{K!}(-1)^K \sum_{\delta=0}^{K} \binom{K}{\delta} (-h_i)^{K-\delta} \frac{(t_{i+1}-u)^{\delta+3-\alpha_\theta}}{\delta+3-\alpha_\theta}$$
$$+\sum_{k=0}^{1}\left(a_{i+1,3+k} \frac{(\mu+k)!}{k!} \frac{(v-t_{i+1})^{3+k-\alpha_\theta}}{3+k-\alpha_\theta} \right) = 0.$$

$$a_{i,3}A_{4,0,3} - 4a_{i,4}A_{4,1,3} + a_{i+1,3}B_{4,0,3} + 4a_{i+1,4}B_{4,1,3} = 0. \quad (3.11)$$

where,

$$A_{4,K,3} = \sum_{\delta=0}^{K} \binom{K}{\delta} (-h_i)^{K-\delta} \frac{(t_{i+1}-u)^{\delta+3-\alpha_\theta}}{\delta+3-\alpha_\theta}, \quad B_{q,k,3} = \frac{(v-t_{i+1})^{\mu+3-\alpha_\theta}}{3+k-\alpha_\theta}.$$
$$K = 0,1, \qquad k = 0,1, \qquad \theta = 1,2,3.$$

The system of eight linear equation (3.7) , (3.8) , (3.10) and (3.11) can be solved to find the spline having fractional continuity α ∈ (2,3).

Similarly for $p = q = 4, \mu = 4$, the equation $a_{i,4} = a_{i+1,4}$ will be added in the set of linear equations (3.8) and (3.10), from (2.10).

$$\sum_{K=0}^{0} a_{i,4+K} \frac{(4+K)!}{K!}(-1)^K A_{4,K,4} + \sum_{k=0}^{q-\mu} a_{i+1,\mu+k} \frac{(\mu+k)!}{k!} B_{4,k,4} = 0.$$
$$a_{i,4}A_{4,0,4} 4a_{i+1,4}B_{4,0,4} = 0 \quad (3.12)$$

Where,

$$A_{4,0,3} = \sum_{\delta=0}^{0} \binom{K}{\delta} (-h_i)^{K-\delta} \frac{(t_{i+1}-u)^{\delta+3-\alpha_\theta}}{\delta+3-\alpha_\theta} = \frac{(t_{i+1}-u)^{4-\alpha_\theta}}{4-\alpha_\theta},$$
$$B_{4,0,4} = \frac{(v-t_{i+1})^{4-\alpha_\theta}}{4-\alpha_\theta}, \qquad \theta = 1.$$

system of eight equation is developed, this system of linear equation does not have unique solution. Infinite solution can be determined by use one free variable. for each value of free variable there is spline that have the fractional continuity of same order. The graphical representation is given in Fig.: 3 (b)-(c).

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3.1.5 For $\boldsymbol{p} = \boldsymbol{q} = \mathbf{5}$ and $\boldsymbol{\mu} = \mathbf{1}$

Similarly, for $p = q = 5, \mu = 1$, in (2.1), (2.2) and (2.7), the following equation are obtained to find the fractional continuity of order $\alpha \in (0,1)$ of polynomial splines of degree 5.

$$P_i(t) = a_{i,0} + a_{i,1}(t-t_i) + a_{i,2}(t-t_i)^2 + a_{i,3}(t-t_i)^3 + a_{i,4}(t-t_i)^4 + a_{i,5}(t-t_i)^5 \tag{3.13}$$

$$P_{i+1}(t) = a_{i+1,0} + a_{i+1,1}(t-t_i) + a_{i+1,2}(t-t_i)^2 + a_{i+1,3}(t-t_i)^3 + a_{i+1,4}(t-t_i)^4 + a_{i+1,5}(t-t_i)^5 \tag{3.14}$$

$$a_{i+1,1}h_{i+1} + a_{i+1,2}h_{i+1}^2 + a_{i+1,3}h_{i+1}^3 + a_{i+1,4}h_{i+1}^4 + a_{i+1,5}h_{i+1}^5 = y_{i+2} - y_{i+1} \tag{3.15}$$

This implies

$$\sum_{K=0}^{4} a_{i,1+K} \frac{(1+K)!}{K!} (-1)^K A_{5,K,1} + \sum_{k=0}^{4} a_{i+1,1+k} \frac{(\mu+k)!}{k!} B_{5,k,1} = 0 \tag{3.16}$$

where

$$A_{5,K,1} = \sum_{\delta=0}^{K} \binom{K}{\delta} (-h_i)^{K-\delta} \frac{(t_{i+1}-u)^{\delta+\mu-\alpha_\theta}}{\delta+\mu-\alpha_\theta}, \quad B_{q,k,\mu} = \frac{(v-t_{i+1})^{\mu+k-\alpha_\theta}}{\mu+k-\alpha_\theta}$$

$$K = 0,1,2,\cdots 4, \qquad k = 0,1,2,\cdots 4, \theta = 1,2,3,\cdots,7.$$

3.1.6 For $\boldsymbol{p} = \boldsymbol{q} = \mathbf{10}$

Now for p = q = 10, the set of equations from proposed algorithm are

$$P_i(t) = \sum_{j=0}^{10} a_{i,j}(t-t_i)^j$$

$$P_{i+1}(t) = \sum_{j=0}^{10} a_{i,j}(t-t_{i+1})^j \tag{3.18}$$

The (2.4) takes the form

$$\sum_{j=0}^{10} a_{i+1,j} h_{i+1}^j = y_{i+2} - y_{i+1} \tag{3.19}$$

From (2.8)

$$\sum_{k=r}^{10} {}^{R}P_r a_{i,R} (t-t_i)^{R-r} = r!\, a_{i+1,r} \tag{3.20}$$

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From (2.10)

$$\sum_{K=0}^{10-\mu} a_{i,\mu+K}\frac{(\mu+K)!}{K!}(-1)^K A_{10,K,\mu} + \sum_{k=0}^{10-\mu} a_{i+1,\mu+k}\frac{(\mu+k)!}{k!}B_{10,k,\mu} = 0 \tag{2.21}$$

$$A_{10,K,\mu} = \sum_{\delta=0}^{K}\binom{K}{\delta}(-h_i)^{K-\delta}\frac{(t_{i+1}-u)^{\delta+\mu-\alpha}}{\delta+\mu-\alpha}, \quad B_{10,k,\mu} = \frac{(v-t_{i+1})^{\mu+k-\alpha}}{\mu+k-\alpha}$$

$$K = 0,1,2,\cdots 10-\mu,$$
$$k = 0,1,2,\cdots 10-\mu,$$
$$\theta = 1,2,3,\cdots,18-\mu.$$

The system of these are 20 equations is consistent and have unique solution for $1 \leq \mu \leq 9$. For $\mu = 10$, the system does not posses unique solution, there are some free variables, so by assigning the suitable different values to these free variable there are infinite many splines of same orders of continuity. The graphs of different values of μ are are shown in the Fig. 5.

3.2 Case-II: ($\boldsymbol{p \neq q}$)

In this case the polynomials spline functions have different degree $p \neq q$ that can be explained by assigning different values in general algorithm consisting the equations.

3.3 For $\boldsymbol{p = 3, q = 4}$ and $\boldsymbol{\mu = 1}$

After putting $p = 3, q = 4$ in (2.2), we get the following splines

$$\begin{aligned} P_i(t) &= a_{i,0} + a_{i,1}(t-t_i) + a_{i,2}(t-t_i)^2 + a_{i,3}(t-t_i)^3 \\ P_{i+1}(t) &= a_{i+1,0} + a_{i+1,1}(t-t_i) + a_{i+1,2}(t-t_i)^2 + a_{i+1,3}(t-t_i)^3 \\ &\quad + a_{i+1,4}(t-t_i)^4 \end{aligned} \tag{3.22}$$

From (2.4), for $q = 4$, obtained (3.7), and for $\mu = 1$, we obtain

$$\begin{aligned} a_{i,0} + a_{i,1}h_i + a_{i,2}h_i^2 + a_{i,3}h_i^3 &= a_{i+1,0} \\ a_{i,1} + 2a_{i,2}h_i + 3a_{i,3}h_i^2 &= a_{i+1,1} \end{aligned} \tag{3.23}$$

The (2.10) takes the form

$$\sum_{K=0}^{2} a_{i,1+K}\frac{(1+K)!}{K!}(-1)^K A_{3,K,1} + \sum_{k=0}^{3} a_{i+1,1+k}\frac{(1+k)!}{k!}B_{4,k,1} = 0, \tag{2.10}$$

where,

$$A_{3,K,1} = \sum_{\delta=0}^{K}\binom{K}{\delta}(-h_i)^{K-\delta}\frac{(t_{i+1}-u)^{\delta+1-\alpha_\theta}}{\delta+1-\alpha_\theta}, \quad B_{4,k,1} = \frac{(v-t_{i+1})^{1+k-\alpha_\theta}}{1+k-\alpha_\theta}$$

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$$K = 0,1,2, \qquad k = 0,1,2,3, \quad \theta = 1,2,3,4.$$

3.4 For $p = 3, q = 4$ and $\mu = 2$

we obtain the set of three equations (3.3), and (2.10) takes the form

$$\sum_{K=0}^{1} a_{i,2+K} \frac{(2+K)!}{K!} (-1)^K A_{3,K,1} + \sum_{k=0}^{2} a_{i+1,2+k} \frac{(2+k)!}{k!} B_{4,k,1} = 0, \tag{2.25}$$

where,

$$A_{3,K,2} = \sum_{\delta=0}^{K} \binom{K}{\delta} (-h_i)^{K-\delta} \frac{(t_{i+1}-u)^{\delta+2-\alpha_\theta}}{\delta+1-\alpha_\theta}, \quad B_{4,k,2} = \frac{(v-t_{i+1})^{2+k-\alpha_\theta}}{2+k-\alpha_\theta}$$

$$K = 0,1, \qquad k = 0,1,2, \quad \theta = 1,2,3.$$

3.5 For $p = 3, q = 4$ and $\mu = 3$

One equation is obtained from (2.4) and the set of equations in obtained from

$$\begin{aligned} a_{i,0} + a_{i,1}h_i + a_{i,2}h_i^2 + a_{i,3}h_i^3 &= a_{i+1,0} \\ a_{i,1} + 2a_{i,2}h_i + 3a_{i,3}h_i^2 &= a_{i+1,1} \\ a_{i,2} + 3a_{i,3}h_i &= a_{i+1,2} \end{aligned} \tag{3.26}$$

The (2.10) takes the form

$$\sum_{K=0}^{0} a_{i,3+K} \frac{(3+K)!}{K!} (-1)^K A_{3,K,3} + \sum_{k=0}^{2} a_{i+1,3+k} \frac{(3+k)!}{k!} B_{4,k,3} = 0, \tag{2.27}$$

where,

$$A_{3,K,3} = \sum_{\delta=0}^{0} \binom{0}{\delta} (-h_i)^{0-\delta} \frac{(t_{i+1}-u)^{\delta+3-\alpha_\theta}}{\delta+3-\alpha_\theta}, \quad B_{4,k,3} = \frac{(v-t_{i+1})^{3+k-\alpha_\theta}}{3+k-\alpha_\theta}$$

$$k = 0,1, \quad \theta = 1,2.$$

After solving the above system of equation, one can get the spline of fractional continuity. The graphical representations for different values of μ is presented in Fig. 6.

3.6 For p = 12, q = 9 and $\mu = 4, 5, 6, 7, 8$ & 9

Similarly, for $p = 12$, $q = 9$ and for $\mu = 4,5,6,7,8$ and 9. The graphical representation is show in Fig. 7.

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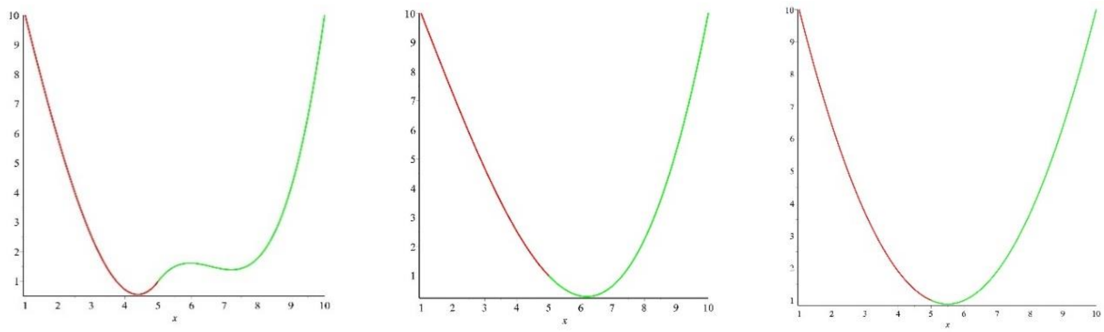
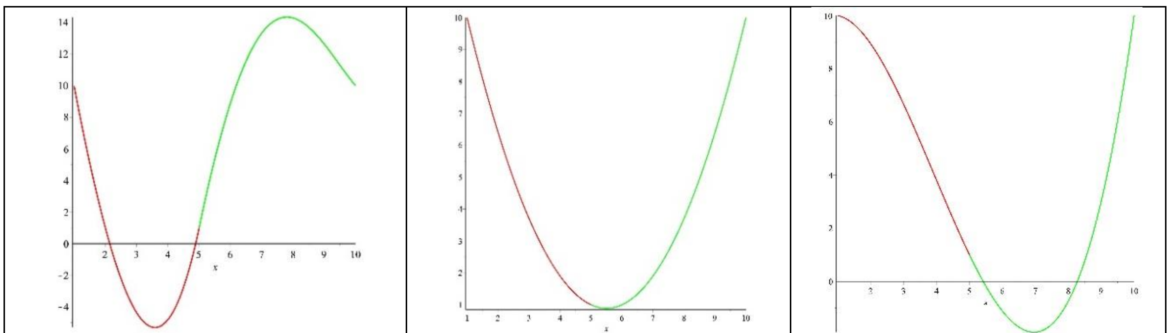


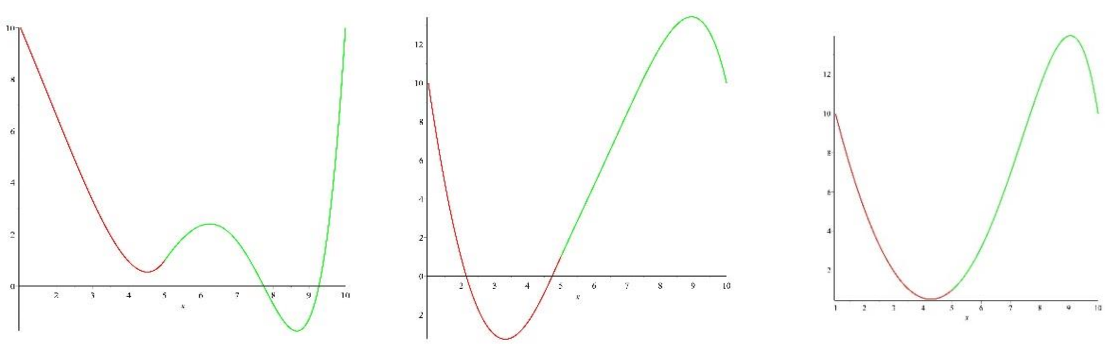
(a) (b) (c)

Figure 2: Represent the fractional continuity of cubic spline of order α (a) For $\mu = 1$, shows the graph of the spline of order $C^{0.85}, C^{0.90}$ and $C^{0.95}$ (b) For $\mu = 2$ shows the graph of the spline of order $C^{1.96}$ and $C^{1.97}$ and in (c) For $\mu = 3$ shows the graph of the spline of order $C^{2.95}$



(a)	(b)	(c)

Figure 3: Represent the fractional continuity of spline of degree 4 of order α (a) For $\mu = 1$, the spline have the level of continuity $C^{0.12}, C^{0.24}, C^{0.36}, C^{0.48}$ and $C^{0.60}$, (b) for $\mu = 3$, $C^{2.6}, C^{2.74}$, and $C^{2.8}$, (c) for $\mu = 4$, $C^{3.3}$ and $C^{3.5}$



(a) (b) (c)

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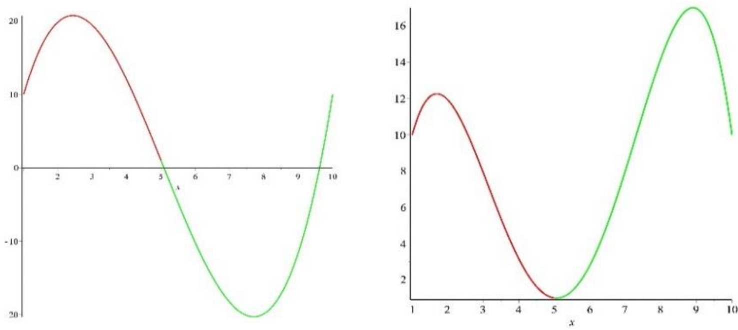
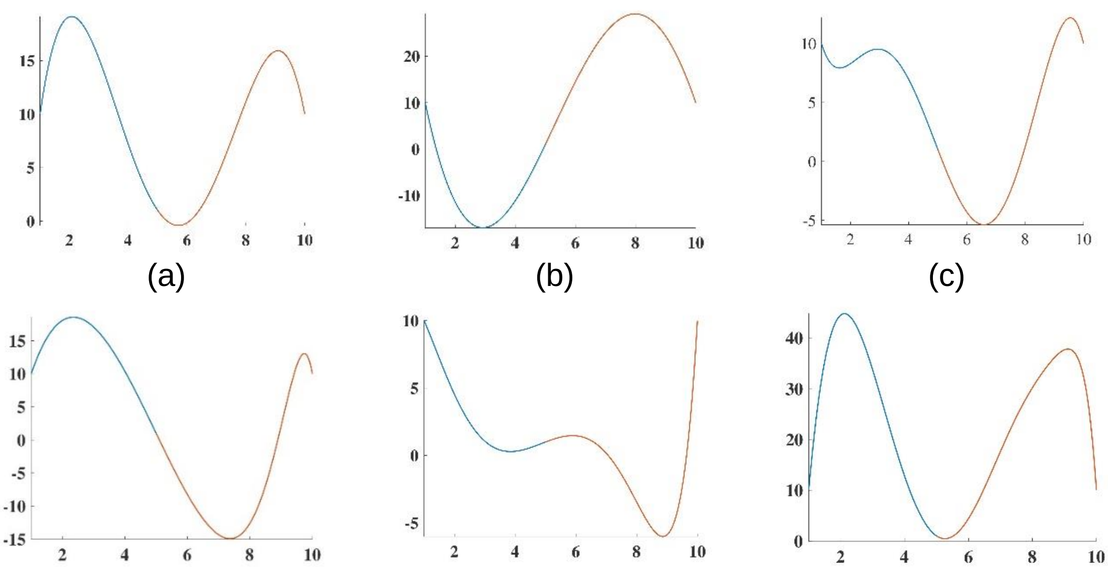


(d) (e)

Figure 4: Represent the fractional continuity of spline of degree 5 of order α (a) For $\mu = 1$, the spline have the level of continuity $C^{0.52}, C^{0.54}, C^{0.56}, C^{0.58}, C^{0.60}, C^{0.62}$ and $C^{0.64}$. (b) For $\mu = 2$, the spline have the levels of continuity $C^{1.2}, C^{1.3}, C^{1.4}, C^{1.5}, C^{1.6}$ and $C^{1.7}$. (c) For $\mu = 3$, the spline have the level of continuity $C^{2.12}, C^{2.14}, C^{2.16}, C^{2.18}$ and $C^{2.20}$. (d) For $\mu = 4$, the spline have the levels of continuity $C^{3.41}, C^{3.42}, C^{3.43}$ and $C^{4.40}$. (e) For $\mu = 5$, the spline have the level of continuity $C^{4.96}, C^{4.97}$ and $C^{4.98}$



(a) (b) (c)

(d) (e) (f)

Figure 5: Represent the fractional continuity of spline of degree 10 of order α (a) For $\mu = 5$, the spline have the level of continuity $C^{4.71}, C^{4.72}, \cdots, C^{4.83}$ (b) For $\mu = 6$, the spline have the level of continuity $C^{5.71}, C^{5.72}, \cdots, C^{5.82}$. (c) For μ = 7, the spline have the level of continuity $C^{6.71}, C^{6.72}, \cdots, C^{6.81}$. (d) For μ = 8, the spline have the level of continuity $C^{7.51}, C^{7.52}, \cdots, C^{7.60}$. (e) For $\mu = 9$, the spline have the levels of continuity $C^{8.51}, C^{8.52}, \cdots, C^{8.59}$. (f) For μ = 10, the spline have the level of continuity $C^{9.51}, C^{9.52}, \cdots, C^{9.58}$

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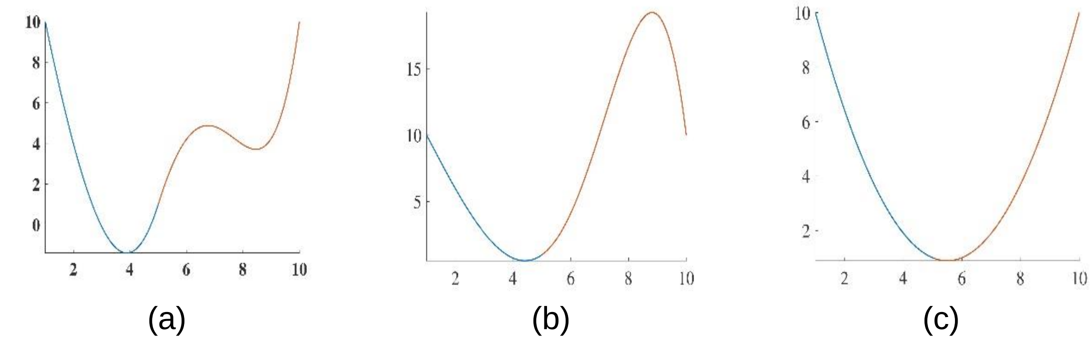

Figure 6: Represent the graphs of the case in which degree of polynomials are not equal here $p = 3, q = 4$, (a) for $\mu = 1$, there four continuity levels, $C^{0.71}, C^{0.72}, C^{0.73}$ and $C^{0.74}$. (b) for $\mu = 2$, there three continuity levels, $C^{1.71}, C^{1.72}$ and $C^{1.73}$. (c) for $\mu = 3$, there two continuity levels, $C^{2.71}$ and $C^{2.72}$

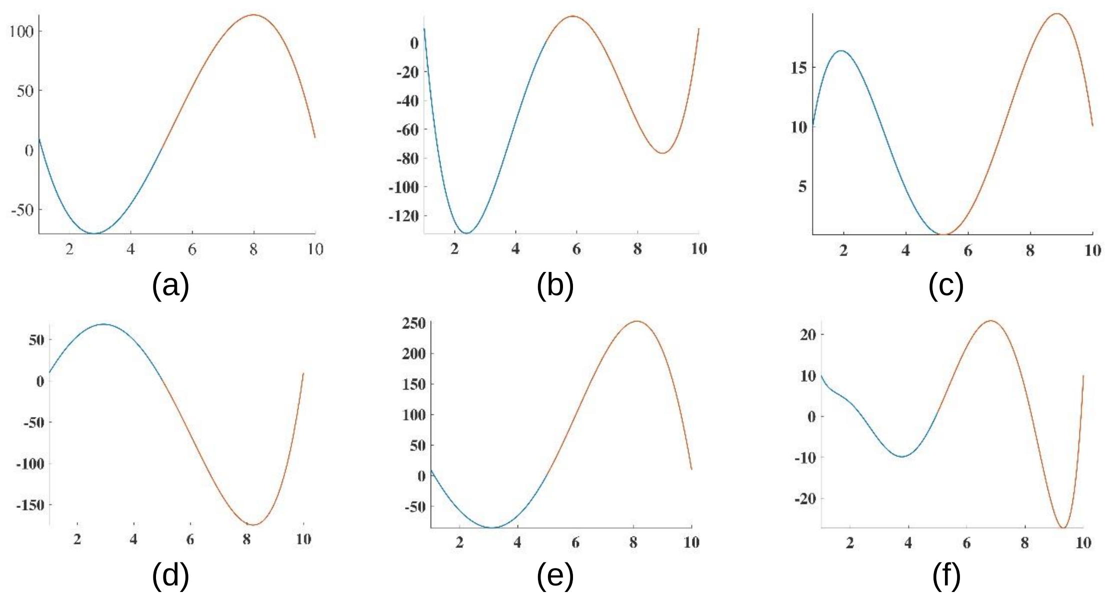

Figure 7: Represent the graphs of the case in which degree of polynomials are not equal here $p = 12, q = 9$, (a) for $\mu = 4$, the continuity levels are, $C^{3.71}, C^{3.72}, \cdots, C^{3.85}$ (b) for $\mu = 5$, the continuity levels are, $C^{4.71}, C^{4.72}, \cdots, C^{4.84}$. (c) for μ = 6, the continuity levels are, $C^{5.71}, C^{5.72}, \cdots, C^{5.83}$. (d) for $\mu = 7$, the continuity levels are, $C^{6.71}, C^{6.72}, \cdots, C^{6.82}$ (e) for $\mu = 8$, the continuity levels are, $C^{7.71}, C^{7.72}, \cdots, C^{7.81}$, (f) for $\mu = 9$, the continuity levels are, $C^{8.71}, C^{8.72}, \cdots, C$^8.80

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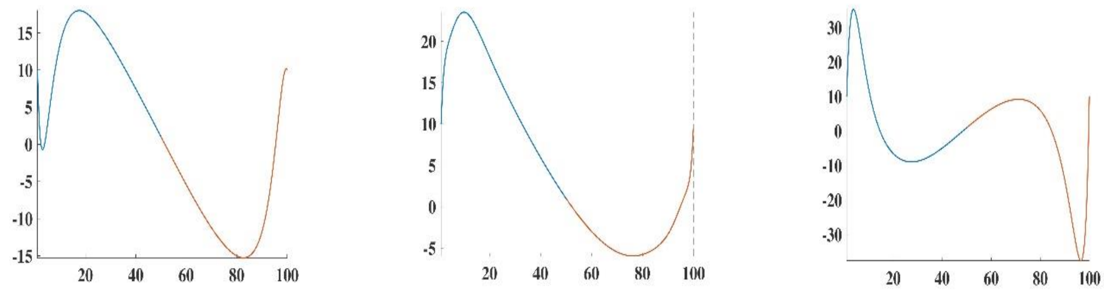

(a) (b) (c)

Figure 8: Represent the graphs of polynomial of degree 50, wide support area is used, in narrow support area free variables are required to find the solution. (a) for μ = 48, the continuity levels are, $C^{47.701}, C^{47.702}, \cdots, C^{47.750}$. (b) for $\mu = 49$, the continuity levels are, $C^{48.701}, C^{48.702}, \cdots, C^{48.749}$. (c) for $\mu = 50$, the continuity levels are, $C^{49.701}, C^{49.702}, \cdots, C^{49.748}$

4. CONCLUSION

The results discussed in this work show the credibility and potential of the recently developed piecewise general fractional order spline technique as an acceptable replacement for older techniques of shape preservation. The study also shows that, we can locally adjust the piecewise curves using the novel technique by changing α or the two introduced parameters u and v, which are used in left hand and right hand Caputo derivative. Further evidence that the piecewise curves will definitely pass over the given data points is provided by this. The ultimate shape of the curve is still largely in our hands.

References

1) Greville, T. N. (1964). Numerical procedures for interpolation by spline functions. Journal of the Society for Industrial and Applied Mathematics, Series B: Numerical Analysis, 1(1), 53-68.

2) Smith, P. L. (1979). Splines as a useful and convenient statistical tool. The American Statistician, 33(2), 57-62.

3) Bo¨hm, W., Farin, G., and Kahmann, J. (1984). A survey of curve and surface methods in CAGD. Computer Aided Geometric Design, 1(1), 1-60.

4) Cox, D. D. (1984). Multivariate smoothing spline functions. SIAM Journal on Numerical Analysis, 21(4), 789-813.

5) Hobby, J. D. (1986). Smooth, easy to compute interpolating splines. Discrete and computational geometry, 1(2), 123-140.

6) DeRose, T. D., and Barsky, B. A. (1988). Geometric continuity, shape parameters, and geometric constructions for catmull-rom splines. ACM Transactions on Graphics (TOG), 7(1), 1-41.

7) Froese Fischer, C., and Idrees, M. (1989). Spline algorithms for continuum functions. Computers in Physics, 3(3), 53-58.

8) Milroy, M. J., Bradley, C., Vickers, G. W., and Weir, D. J. (1995). G1 continuity of B-spline surface patches in reverse engineering. Computer-Aided Design, 27(6), 471-478.

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E-Publication: Online Open Access
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DOI: 10.5281/zenodo.13898316

9) Zahra, W. K., and Elkholy, S. M. (2012). The use of cubic splines in the numerical solution of fractional differential equations. International Journal of Mathematics and Mathematical Sciences, 2012.

10) Pepin, A., Beauchemin, S. S., L´eger, S., and Beaudoin, N. (2020). A new method for high-degree spline interpolation: proof of continuity for piecewise polynomials. Canadian Mathematical Bulletin, 63(3), 655-669.

11) He, L., Zhu, Y., and Lu, Z. (2024). A cubic spline interpolation based numerical method for fractional differential equations. Journal of Industrial and Management Optimization, 20(8), 2770-2794.

12) Luchko, Y., and Trujillo, J. (2007). Caputo-type modification of the Erd´elyiKober fractional derivative. Fractional Calculus and Applied Analysis, 10(3), 249-267.

13) Machado, J. T., Kiryakova, V., and Mainardi, F. (2011). Recent history of fractional calculus. Communications in nonlinear science and numerical simulation, 16(3), 1140-1153.

14) Frederico, G. S., and Lazo, M. J. (2016). Fractional Noether's theorem with classical and Caputo derivatives: constants of motion for non-conservative systems. Nonlinear Dynamics, 85, 839-851.

15) Brzezin´ski, D. W. (2017). Comparison of fractional order derivatives computational accuracy-right hand vs left hand definition. Applied Mathematics and Nonlinear Sciences, 2(1), 237-248.

16) Rehman, H. U., Darus, M., and Salah, J. (2018). A note on Caputo's derivative operator interpretation in economy. Journal of Applied Mathematics, 2018.

17) Kirmani, S., binti Mohd Suaib, N., and Bilal Riaz, M. (2019). Shape preserving fractional order KNR C^1 cubic spline. The European Physical Journal Plus, 134, 1-8.

18) Kirmani, S. K. N., Riaz, M. B., Jarad, F., Jasim, H. N., and Enver, A. (2021). Shape Preserving Piecewise KNR Fractional Order Biquadratic C^2 Spline. Journal of Mathematics, 2021, 1-9.

19) Pratap, H., Kumar, S., and Singh, G. (2024). Brief History of Fractional Calculus: A Survey. Migration Letters, 21(S7), 238-243.