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# POISSON-MODIFIED MISHRA DISTRIBUTION

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### Abstract

This is a compound probability distribution based on a single parameter and it has been obtained by compounding Poisson distribution with Modified Mishra distribution of Sah (2022). The proposed distribution is named as Poisson-Modified Mishra Distribution (PMMD). The essential characteristics of the proposed distribution have been obtained. The method of moments and the method of maximum likelihood have been discussed to obtain an estimator of the proposed distribution. Applying goodness of fit to some over-dispersed data sets for the verification of the validity of the theoretical work, we conclude that the proposed distribution would be a better alternative of Poisson-Lindley distribution (PLD) of Sankaran (1970) and Poisson-Mishra distribution (PMD) of Sah (2017).

**Keywords:** Poisson-Modified Mishra distribution, Poisson-Lindley distribution, Modified Mishra distribution, Moments, Estimation, Mixing.

## INTRODUCTION:

It can be found that many research papers have been published in the sector of compound probability distribution. Poisson distribution is also called a distribution of rare events. Many research papers have been published in the field of countable and continuous mixtures of Poisson distribution since the last three decades. The main purpose of proposing this distribution is for the better modelling of over-dispersed discrete data-sets than previously obtained continuous mixtures of Poisson distribution based on a single parameter by others. To derive Poisson-Modified Mishra distribution (PMMD), Modified Mishra distribution (MMD) (Sah, 2022) is compounded with Poisson distribution, where the parameter of the Poisson distribution follows the MMD given by its probability density function

$$f_1(y;\alpha)=\frac{\alpha^3}{(2+\alpha+\pi\alpha^2)}(\pi+y+y^2)e^{-\alpha y}\ ;y>0,\alpha>0 \tag{1}$$

MMD (1) is a modified form of Mishra distribution (Sah, 2015) given by its density function

$$f_2(y;\alpha)=\frac{\alpha^3}{(2+\alpha+\alpha^2)}(1+y+y^2)e^{-\alpha y}\ ;y>0,\alpha>0 \tag{2}$$

Lindley mixture of Poisson distribution results in PLD obtained by Sankaran, (Sankaran, 1970) and it can be observed that this distribution is also based on a single parameter.

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For this reason, it seems appropriate to compare the results of goodness of fit obtained by using PMMD and PLD. Probability density function of Lindley distribution (Lindley, 1958) was given by

$$f_3(y;\alpha)=\frac{\alpha^2}{(1+\alpha)}(1+y)e^{-\alpha y}\ ;y>0,\alpha>0 \qquad (3)$$

The expression (1), (2) and (3) are based on a single parameter and all are continuous in nature. Hence, it is the appropriate reason to compare the proposed distribution with PLD and PMD. Probability mass function of PLD and PMD are given by the expression (4) and (5) respectively.

$$P_4(Y;\alpha)=\frac{\alpha^2(y+\alpha+2)}{(\alpha+1)^{y+3}}\quad ;y=0,1,2,\ldots\quad ;\alpha>0 \qquad (4)$$

It was introduced by Sankaran (1970) to model count data. The Probability mass function of PMD was given by

$$P_5(Y;\alpha)=\frac{\alpha^3[(1+\alpha)(y+\alpha+2)+(1+y)(2+y)]}{(\alpha^2+\alpha+1)(1+\alpha)^{y+3}}\quad ;y=0,1,2,\ldots\quad ;\alpha>0 \qquad (5)$$

It has been obtained by mixing Poisson distribution with Mishra distribution (Sah, 2015) and it has been observed that PMD (5) and Generalised Poisson-Mishra (Sah, 2018) both are very useful to study count data-sets.

This paper has been classified under different topics to give it shape. Introduction part is placed in the first section of this paper. The materials and methods for this paper are presented in the second section. The results obtained in this paper are placed in the third section and the results obtained are placed in following different sub-headings for simplicity.

- Poisson-Modified Mishra distribution and Related Characteristics of PMMD.
- Statistical Moments and Some Descriptive Measures of Statistics of PMMD.
- Methods of Estimation of the Parameter of PMMD, and
- Goodness of Fit and Applications of PMMD.

Conclusion about this paper has been placed in the last section.

In brief, information about this paper can be expressed as follows. This distribution is started with the construction of its probability mass function, which is obtained by compounding the Poisson distribution with MMD. In order to know about the variability, shape and size of this distribution, the first four moments about the origin and moments about the mean have been discussed and obtained. A good estimator of the parameter seems absolutely necessary to apply the goodness of fit, for that two methods of estimation have been used. All the data used for the application purpose of this paper are secondary in nature and has been has been included in references as a courtesy to its authors. It has been found that P-value for PMMD is greater than the P-value of PLD

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and PMD and so, it is concluded that PMMD would be a better alternative of PMD as well as PLD for the similar nature of data-sets.

## MATERIAL AND METHODS:

This paper begins with construction of theoretical works and end with its applications along with concluding remarks. Probability mass function of PMMD has been constructed by mixing Poisson distribution with MMD (Sah, 2022). The method of moments and the method of maximum likelihood have been used to estimate the parameters of this distribution. Goodness of fit has been applied to some over-dispersed secondary count data-sets to check the validity of the theoretical work and at the end, we conclude that PMMD is a better alternative to the PLD as well as MMD.

## RESULTS

The results obtained of this distribution are presented sequentially under following different sub-headings.

- Poisson-Modified Mishra distribution and Related Characteristics of PMMD.
- Statistical Moments and Some Descriptive Measures of Statistics of PMMD.
- Methods of Estimation of the Parameter of PMMD, and
- Goodness of Fit and Applications of PMMD.

### 1. Poisson-Modified Mishra distribution and Related Characteristics of PMMD:

Let $\alpha$ and $\lambda$ the parameters of the MMD and Poisson distribution respectively. The proposed distribution can be generated by mixing Poisson distribution with MMD where $\lambda$ is continuous in nature and it follows MMD. It gives Probability mass function of PMMD. It is given by the expression (6) as follows.

$$P(Y;\alpha)=\int_0^\infty\left[\left(\frac{e^{-\lambda}\lambda^y}{y!}\right)\left(\frac{\alpha^3}{(2+\alpha+\pi\alpha^2)}\right)(\pi+\lambda+\lambda^2)e^{-\alpha\lambda}\right]d\lambda\ \ ;y=0,1,2,\ldots;\lambda>0;\alpha>0$$

$$=\left(\frac{\alpha^3}{(2+\alpha+\pi\alpha^2)}\frac{1}{y!}\right)\left[\pi\int_0^\infty\lambda^y e^{-(1+\alpha)\lambda}d\lambda+\int_0^\infty\lambda^{y+1}e^{-(1+\alpha)\lambda}d\lambda+\int_0^\infty\lambda^{y+2}e^{-(1+\alpha)\lambda}d\lambda\right]$$

$$=\left(\frac{\alpha^3}{(2+\alpha+\pi\alpha^2)}\right)\left[\frac{(1+\alpha)(1+\pi+\pi\alpha+y)+(1+y)(2+y)}{(1+\alpha)^{y+3}}\right];y=0,1,2,\ldots;\alpha>0 \qquad (6)$$

Graphical presentations of p.m.f. of PMMD (6) at varying values of parameter ($\alpha$) are given as

**Fig 1: Graph of p.m.f. of PMMD at** $\alpha = 0.5, 1.0, 1.5, 2.0$

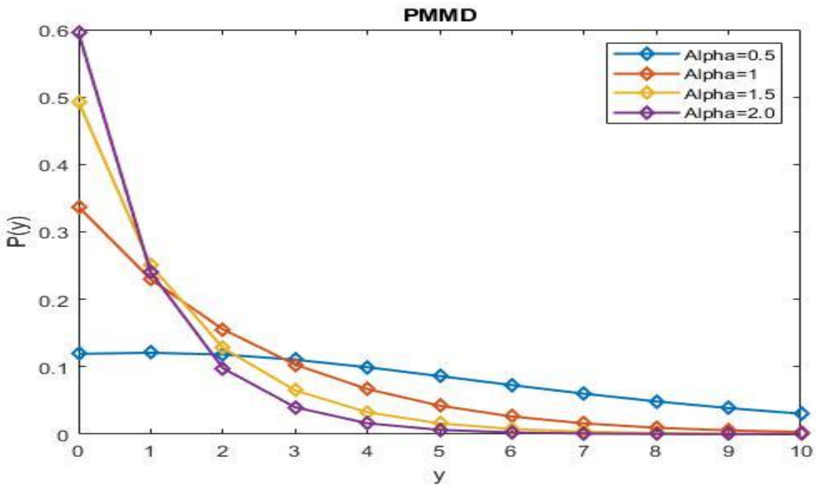



**Fig 2: Graph of p.m.f. of PMMD at** $\alpha = 2.0, 3.0, 4.0, 5.0$

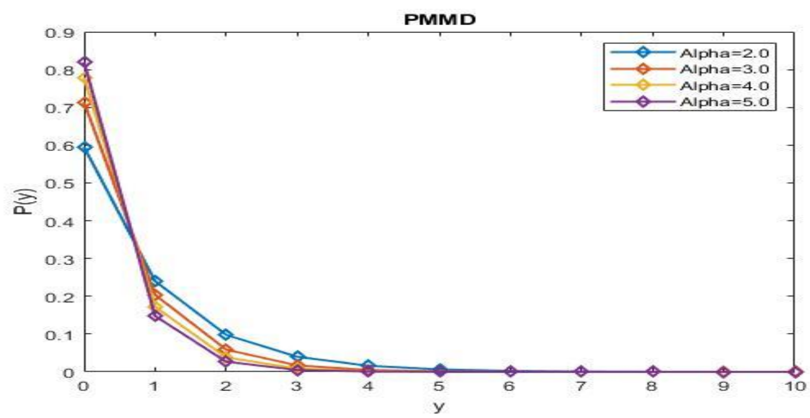
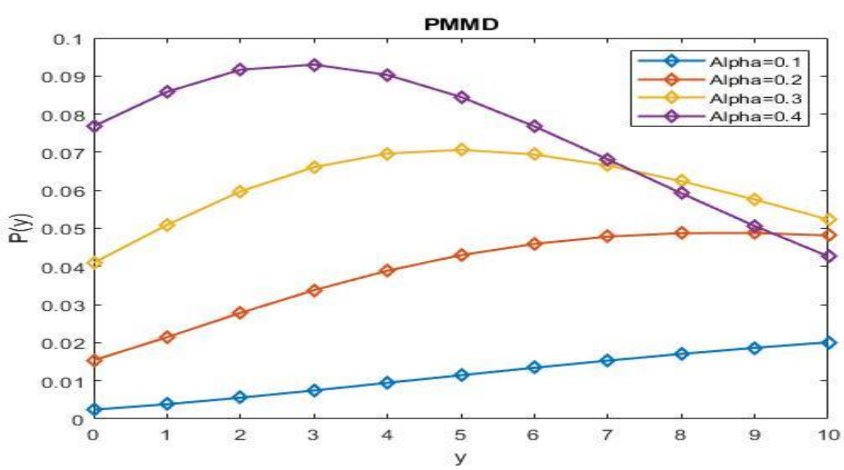



**Fig.3 Graph of p.m.f. of PMMD a** $\alpha = 0.1, 0.2, 0.3, 0.4$



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From the above figures, it has been observed that as the value of $\alpha$ increases, value of $f(y)$ also increases at $y=0$. It can also be observed that the $f(y)$ curve touches X-axis first having greater value of $\alpha$.

Probability Generating Function (p.g.f.) of PMMD: It is given by the expression (7) and can be obtained as follows.

$$P_Y^{(t)} = \frac{\alpha^3}{(2+\alpha+\pi\alpha^2)}\int_0^\infty e^{-(1-t)\lambda}(\pi+\lambda+\lambda^2)e^{-\alpha\lambda}d\lambda$$

$$= \frac{\alpha^3}{(2+\alpha+\pi\alpha^2)}\int_0^\infty (\pi+\lambda+\lambda^2)e^{-(1+\alpha-t)\lambda}d\lambda$$

$$= \frac{\alpha^3}{(2+\alpha+\pi\alpha^2)}\left[\pi\int_0^\infty e^{-\lambda(1+\alpha-t)}d\lambda + \int_0^\infty \lambda e^{-\lambda(1+\alpha-t)}d\lambda + \int_0^\infty \lambda e^{-\lambda(1+\alpha-t)}d\lambda\right]$$

$$= \left(\frac{\alpha^3}{(2+\alpha+\pi\alpha^2)}\right)\left[\frac{\pi(1+\alpha-t)+(1+\alpha-t)+2}{(1+\alpha-t)^3}\right];\alpha>0 \tag{7}$$

Moment Generating Function (M.G.F.) of PMMD: It has significant property to generate the first four moments about the origin and can be obtained as

$$M_Y^{(t)} = \frac{\alpha^3}{(2+\alpha+\pi\alpha^2)}\int_0^\infty (\pi+\lambda+\lambda^2)e^{-\lambda(1+\alpha-e^t)}d\lambda$$

$$= \frac{\alpha^2}{(1+\pi\alpha)}\left[\frac{\pi\Gamma 1}{(1+\alpha-e^t)^1} + \frac{\Gamma 2}{(1+\alpha-e^t)^2} + \frac{\Gamma 3}{(1+\alpha-e^t)^3}\right]$$

$$= \left(\frac{\alpha^3}{(2+\alpha+\pi\alpha^2}\right)\left[\frac{\pi(1+\alpha-e^t)^2+(1+\alpha-e^t)+2}{(1+\alpha-e^t)^3}\right] \tag{8}$$

The first four moments about origin can also be obtained by using the following expression.

$$\mu_r' = \left[\frac{\partial^r[M_Y^{(t)}]}{\partial t^r}\right]_{t=0} \tag{9}$$

## 2. Statistical Moments and Some Descriptive Measures of Statistics of PMMD:

Statistical moments are essential to know about shape, size and variability of any statistical distribution. For that reason, the first four moments about origin as well as about the mean of this distribution have been obtained. The r$^{th}$ moment about origin of PMMD can be obtained as

$$\mu_r' = E\left[E(Y^r/\lambda\right] = \frac{\alpha^3}{(2+\alpha+\pi\alpha^2)}\int_0^\infty\left(\sum_{y=0}^\infty \frac{y^r e^{-\lambda}\lambda^y}{y!}\right)(\pi+\lambda+\lambda^2)e^{-\alpha\lambda}d\lambda \tag{10}$$

Substituting $r=1,2,3,4$ in the equation (10), we can obtain the 1$^{st}$ four moments about origin of PNLED as follows. The mean of PMMD is obtained as

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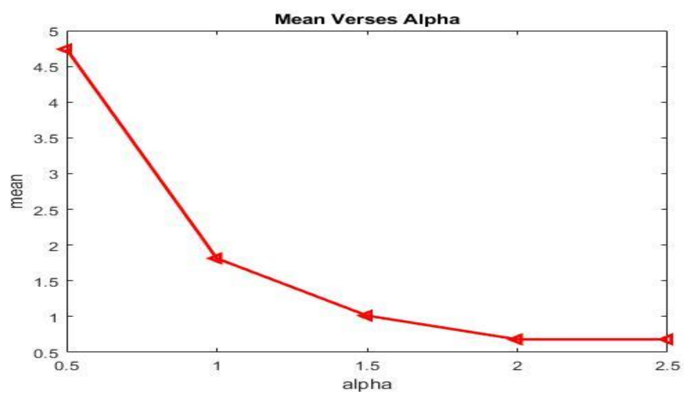
$$\mu_1' = \frac{\alpha^3}{(2+\alpha+\pi\alpha^2)}\int_0^\infty\left(\sum_{y=0}^{\infty}\frac{y^1 e^{-\lambda}\lambda^y}{y!}\right)(\pi+\lambda+\lambda^2)e^{-\alpha\lambda}d\lambda = \frac{\alpha^3}{(2+\alpha+\pi\alpha^2)}\int_0^\infty(\lambda)(\pi+\lambda+\lambda^2)e^{-\alpha\lambda}d\lambda$$

$$= \frac{\alpha^3}{(2+\alpha+\pi\alpha^2)}\left[\int_0^\infty \pi\lambda e^{-\alpha\lambda}d\lambda + \int_0^\infty \lambda^2 e^{-\alpha\lambda}d\lambda + \int_0^\infty \lambda^3 e^{-\alpha\lambda}d\lambda\right]$$

$$= \frac{\alpha^3}{(2+\alpha+\pi\alpha^2)}\left[\frac{\pi}{\alpha^2}+\frac{2}{\alpha^3}+\frac{6}{\alpha^4}\right] = \frac{(6+2\alpha+\pi\alpha^2)}{\alpha(2+\alpha+\pi\alpha^2)} \qquad (11)$$

Graphical representation of the mean for varying values of alpha is given below.

**Fig.4 The mean of PMMD for** $\alpha = 0.5, 1.0, 1.5, 2.0, 2.5$




From figure (4), we can see that the mean of PMMD decreases as the value of the parameter increases. Substituting $r=2$ in the equation (10), the second moment about the origin of PMMD has been obtained as follows

$$\mu_2' = \frac{\alpha^3}{(2+\alpha+\pi\alpha^2)}\int_0^\infty\left(\sum_{y=0}^{\infty}\frac{y^2 e^{-\lambda}\lambda^y}{y!}\right)(\pi+\lambda+\lambda^2)e^{-\alpha\lambda}d\lambda = \frac{\alpha^3}{(2+\alpha+\pi\alpha^2)}\int_0^\infty(\lambda+\lambda^2)(\pi+\lambda+\lambda^2)e^{-\alpha\lambda}d\lambda$$

$$= \frac{\alpha^3}{(2+\alpha+\pi\alpha^2)}\left[\left(\frac{\pi}{\alpha^2}+\frac{2}{\alpha^3}+\frac{6}{\alpha^4}\right)+\left(\frac{2\pi}{\alpha^3}+\frac{6}{\alpha^4}+\frac{24}{\alpha^5}\right)\right] = \frac{[\alpha(\pi\alpha+2)+6]}{\alpha(2+\alpha+\pi\alpha^2)}+\frac{[2\alpha(\pi\alpha+3)+24]}{\alpha^2(2+\alpha+\pi\alpha^2)} \qquad (12)$$

Putting $r=3$ in the equation (10), the third moment about the origin can be obtained as

$$\mu_3' = \frac{\alpha^3}{(2+\alpha+\pi\alpha^2)}\int_0^\infty\left(\sum_{y=0}^{\infty}\frac{y^3 e^{-\lambda}\lambda^y}{y!}\right)(\pi+\lambda+\lambda^2)e^{-\alpha\lambda}d\lambda = \frac{\alpha^3}{(2+\alpha+\pi\alpha^2)}\int_0^\infty(\lambda^3+3\lambda^2+\lambda)(\pi+\lambda+\lambda^2)e^{-\alpha\lambda}d\lambda$$

$$= \frac{\alpha^3}{(2+\alpha+\pi\alpha^2)}\int_0^\infty\left(\pi\lambda+3\pi\lambda^2+\pi\lambda^3+\lambda^2+4\lambda^3+4\lambda^4+\lambda^5\right)e^{-\alpha\lambda}d\lambda$$

$$= \frac{\alpha^3}{(2+\alpha+\pi\alpha^2)}\left[\left(\frac{\pi}{\alpha^2}+\frac{2}{\alpha^3}+\frac{6}{\alpha^4}\right)+\left(\frac{6\pi}{\alpha^3}+\frac{18}{\alpha^4}+\frac{72}{\alpha^5}\right)+\left(\frac{6\pi}{\alpha^4}+\frac{24}{\alpha^5}+\frac{120}{\alpha^6}\right)\right]$$

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$$=\frac{[\alpha(\pi\alpha+2)+6]}{\alpha(2+\alpha+\pi\alpha^2)}+\frac{[6\alpha(\pi\alpha+3)+72]}{\alpha^2(2+\alpha+\pi\alpha^2)}+\frac{[6\alpha(\pi\alpha+4)+120]}{\alpha^3(2+\alpha+\pi\alpha^2)} \tag{13}$$

Putting $r=4$ in the equation (10), the fourth moment about origin has been obtained as

$$\mu_4'=\frac{\alpha^3}{(2+\alpha+\pi\alpha^2)}\int_0^\infty\left(\sum_{y=0}^{\infty}\frac{y^4e^{-\lambda}\lambda^y}{y!}\right)(\pi+\lambda+\lambda^2)e^{-\alpha\lambda}d\lambda$$

$$=\frac{\alpha^3}{(2+\alpha+\pi\alpha^2)}\int_0^\infty\left(\lambda^4+6\lambda^3+7\lambda^2+\lambda\right)(\pi+\lambda+\lambda^2)e^{-\alpha\lambda}d\lambda$$

$$=\frac{\alpha^3}{(2+\alpha+\pi\alpha^2)}\int_0^\infty\left(\pi\lambda+7\pi\lambda^2+6\pi\lambda^3+\pi\lambda^4+\lambda^2+8\lambda^3+13\lambda^4+7\lambda^5+\lambda^6\right)e^{-\alpha\lambda}d\lambda$$

$$=\frac{\alpha^3}{(2+\alpha+\pi\alpha^2)}\left[\left(\frac{\pi}{\alpha^2}+\frac{2}{\alpha^3}+\frac{6}{\alpha^4}\right)+\left(\frac{14\pi}{\alpha^3}+\frac{42}{\alpha^4}+\frac{168}{\alpha^5}\right)+\left(\frac{36\pi}{\alpha^4}+\frac{144}{\alpha^5}+\frac{720}{\alpha^6}\right)+\left(\frac{24\pi}{\alpha^5}+\frac{120}{\alpha^6}+\frac{720}{\alpha^7}\right)\right]$$

$$=\frac{\alpha(\pi\alpha+2)+6}{\alpha(2+\alpha+\pi\alpha^2)}+\frac{14(\alpha(\pi\alpha+3)+12)}{\alpha^2(2+\alpha+\pi\alpha^2)}+\frac{36(\alpha(\pi\alpha+4)+20)}{\alpha^3(2+\alpha+\pi\alpha^2)}+\frac{24(\alpha(\pi\alpha+5)+30)}{\alpha^4(2+\alpha+\pi\alpha^2)} \tag{14}$$

**Central Moments of PMMD:**

It is very useful to know about nature of variability, shape and size of PMMD. So, the first four central moments of PMMD can be obtained as

$$\mu_1=0$$

$$\mu_2=\mu_2'-\left(\mu_1'\right)^2=\frac{[\alpha(\pi\alpha+2)+6]}{\alpha(2+\alpha+\pi\alpha^2)}+\frac{[2\alpha(\pi\alpha+3)+24]}{\alpha^2(2+\alpha+\pi\alpha^2)}-\left(\frac{(6+2\alpha+\pi\alpha^2)}{\alpha(2+\alpha+\pi\alpha^2)}\right)^2$$

$$=\frac{[\pi^2\alpha^5+(\pi^2+3\pi)\alpha^4+(12\pi+2)\alpha^3+(16\pi+12)\alpha^2+24\alpha+12]}{[\alpha(\pi\alpha^2+\alpha+2)]^2} \tag{15}$$

Theorem (1): PMMD is an over-dispersed distribution.

Proof:

A distribution is said to be over-dispersed if

Variance > Mean

Or, $\frac{[\pi^2\alpha^5+(\pi^2+3\pi)\alpha^4+(12\pi+2)\alpha^3+(16\pi+12)\alpha^2+24\alpha+12]}{[\alpha(\pi\alpha^2+\alpha+2)]^2}>\frac{(6+2\alpha+\pi\alpha^2)}{\alpha(2+\alpha+\pi\alpha^2)}$

Or, $[\pi^2\alpha^5+(\pi^2+3\pi)\alpha^4+(12\pi+2)\alpha^3+(16\pi+12)\alpha^2+24\alpha+12]-\alpha(6+2\alpha+\pi\alpha^2)(2+\alpha+\pi\alpha^2)>0$ (16)

Which is true because $\alpha>0$ and $\pi=22/7$. Hence, PMMD is an over-dispersed.

Graphical representation of variance of PNLED with varying values of $\alpha$ is given below.

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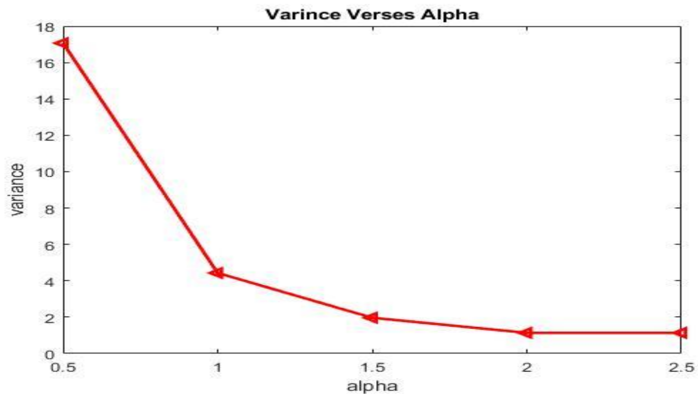
**Fig.5 The Variance of PMMD for** $\alpha = 0.5, 1.0, 1.5, 2.0, 2.5$




From the figure (5), It can be observed that the variance of PMMD decreases as the value of the parameter of the PMMD increases.

The third central moment of PMMD can be obtained as follows

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$$

$$= \left[\frac{[\alpha(\pi\alpha+2)+6]}{\alpha(2+\alpha+\pi\alpha^2)} + \frac{[6\alpha(\pi\alpha+3)+72]}{\alpha^2(2+\alpha+\pi\alpha^2)} + \frac{[6\alpha(\pi\alpha+4)+120]}{\alpha^3(2+\alpha+\pi\alpha^2)}\right] - 3\left[\frac{[\alpha(\pi\alpha+2)+6]}{\alpha(2+\alpha+\pi\alpha^2)} + \frac{[2\alpha(\pi\alpha+3)+24]}{\alpha^2(2+\alpha+\pi\alpha^2)}\right]$$

$$\left[\frac{(6+2\alpha+\pi\alpha^2)}{\alpha(2+\alpha+\pi\alpha^2)}\right] + 2\left[\frac{(6+2\alpha+\pi\alpha^2)}{\alpha(2+\alpha+\pi\alpha^2)}\right]^3$$

$$= \frac{\begin{array}{l}[(\pi^3\alpha^8 + 3\pi^3\alpha^7 + 2\pi^3\alpha^6 + 4\pi^2\alpha^7 + 25\pi^2\alpha^6 + 66\pi^2\alpha^5 + 60\pi^2\alpha^4 + 5\pi\alpha^6 + 42\pi\alpha^5 + 148\pi\alpha^4 \\ 216\pi\alpha^3 + 72\pi\alpha^2 + 2\alpha^5 + 20\alpha^4 + 84\alpha^3 + 158\alpha^2 + 144\alpha + 48)]\end{array}}{[\alpha(2+\alpha+\pi\alpha^2)]^3} \quad (17)$$

The fourth central moment of PMMD has been obtained as

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 + 3(\mu_1')^4$$

$$= \left[\frac{\alpha(\pi\alpha+2)+6}{\alpha(2+\alpha+\pi\alpha^2)} + \frac{14(\alpha(\pi\alpha+3)+12)}{\alpha^2(2+\alpha+\pi\alpha^2)} + \frac{36(\alpha(\pi\alpha+4)+20)}{\alpha^3(2+\alpha+\pi\alpha^2)} + \frac{24(\alpha(\pi\alpha+5)+30)}{\alpha^4(2+\alpha+\pi\alpha^2)}\right]$$

$$-4\left[\frac{[\alpha(\pi\alpha+2)+6]}{\alpha(2+\alpha+\pi\alpha^2)} + \frac{[6\alpha(\pi\alpha+3)+72]}{\alpha^2(2+\alpha+\pi\alpha^2)} + \frac{[6\alpha(\pi\alpha+4)+120]}{\alpha^3(2+\alpha+\pi\alpha^2)}\right]\left[\frac{(6+2\alpha+\pi\alpha^2)}{\alpha(2+\alpha+\pi\alpha^2)}\right]$$

$$+6\left[\frac{[\alpha(\pi\alpha+2)+6]}{\alpha(2+\alpha+\pi\alpha^2)} + \frac{[2\alpha(\pi\alpha+3)+24]}{\alpha^2(2+\alpha+\pi\alpha^2)}\right]\left[\frac{(6+2\alpha+\pi\alpha^2)}{\alpha(2+\alpha+\pi\alpha^2)}\right]^2 - 3\left[\frac{(6+2\alpha+\pi\alpha^2)}{\alpha(2+\alpha+\pi\alpha^2)}\right]^4$$

$$= \frac{\begin{array}{l}[\pi^4\alpha^8(\alpha^3 + 10\alpha^2 + 18\alpha + 9) + \pi^3\alpha^6(5\alpha^4 + 72\alpha^3 + 214\alpha^2 + 648\alpha + 384) \\ +\pi^2\alpha^4(9\alpha^5 + 158\alpha^4 + 640\alpha^3 + 1940\alpha^2 + 5170\alpha + 1224) \\ +\pi\alpha^2(7\alpha^6 + 140\alpha^5 + 712\alpha^4 + 2344\alpha^3 + 6636\alpha^2 + 13480\alpha + 1728) \\ +(2\alpha^7 + 44\alpha^6 + 304\alpha^5 + 3504\alpha^4 + 4464\alpha^3 + 12008\alpha^2 + 11196\alpha + 720)\end{array}}{[\alpha(2+\alpha+\pi\alpha^2)]^4} \quad (18)$$

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The nature of PMMD according to shape and size can be analysed by obtaining the co-efficient of skewness and kurtosis based on moments as follows.

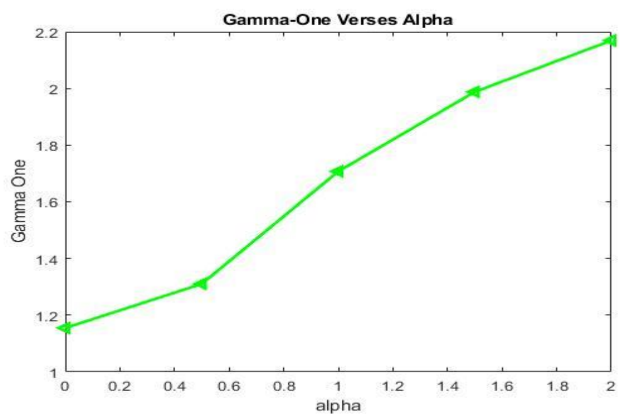
$$\gamma_1 = \frac{\mu_3}{(\mu_2)^{3/2}}$$

$$= \frac{[[(\pi^3\alpha^8 + 3\pi^3\alpha^7 + 2\pi^3\alpha^6 + 4\pi^2\alpha^7 + 25\pi^2\alpha^6 + 66\pi^2\alpha^5 + 60\pi^2\alpha^4 + 5\pi\alpha^6 + 42\pi\alpha^5 + 148\pi\alpha^4 \; 216\pi\alpha^3 + 72\pi\alpha^2 + 2\alpha^5 + 20\alpha^4 + 84\alpha^3 + 158\alpha^2 + 144\alpha + 48)]]}{[\pi^2\alpha^5 + (\pi^2 + 3\pi)\alpha^4 + (12\pi + 2)\alpha^3 + (16\pi + 12)\alpha^2 + 24\alpha + 12]^{3/2}} \qquad (19)$$

The expression (19) co-efficient of skewness of PMMD based on moments. From this expression, it has been observed that $(2/\sqrt{3}) < \gamma_1 < \infty$.

Graphical representation of $\gamma_1$ with different values of the parameter $\alpha$ is given below.

**Fig.6: Skewness of PMMD for** $\alpha = 0.0, 0.5, 1.0, 1.5, 2.0$




$$\beta_2 = \frac{\mu_4}{(\mu_2)^2}$$

$$= \frac{\begin{array}{l}[\pi^4\alpha^8(\alpha^3 + 10\alpha^2 + 18\alpha + 9) + \pi^3\alpha^6(5\alpha^4 + 72\alpha^3 + 214\alpha^2 + 648\alpha + 384) \\ +\pi^2\alpha^4(9\alpha^5 + 158\alpha^4 + 640\alpha^3 + 1940\alpha^2 + 5170\alpha + 1224) \\ +\pi\alpha^2(7\alpha^6 + 140\alpha^5 + 712\alpha^4 + 2344\alpha^3 + 6636\alpha^2 + 13480\alpha + 1728) \\ +(2\alpha^7 + 44\alpha^6 + 304\alpha^5 + 3504\alpha^4 + 4464\alpha^3 + 12008\alpha^2 + 11196\alpha + 720)\end{array}}{[\pi^2\alpha^5 + (\pi^2 + 3\pi)\alpha^4 + (12\pi + 2)\alpha^3 + (16\pi + 12)\alpha^2 + 24\alpha + 12]^2} \qquad (20)$$

The expression (20) is the co-efficient of kurtosis based on moments and it can be noted that $5 < \beta_2 < \infty$.Hence, this distribution is leptokurtic by size. The graphical representation of $\beta_2$ for different values of $\alpha$ is given below

**Jilin Daxue Xuebao (Gongxueban)/Journal of Jilin University (Engineering and Technology Edition)**
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Vol: 42 Issue: 01-2023
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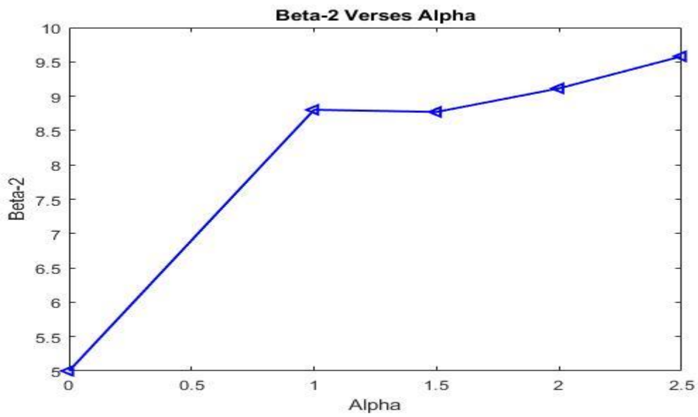
**Fig.7: Kurtosis of PMMD for** $\alpha = 0.0, 1.0, 1.5, 2.0, 2.5$




From figure (6), it can also be noted that $(2/\sqrt{3}) < \gamma_1 < \infty$. It can also be observed that value of $\gamma_1$ increases as the value of $\alpha$ increases. From figure (7), it can also be noted that $5 < \beta_2 < \infty$. It can also be observed that value of $\beta_2$ increases as the value of $\alpha$ increases, while the values of mean and variance are inversely proportional to the value of $\alpha$.

- Remarks:
  - PMMD is always over-dispersed.
  - It is always positively skewed.
  - It is always leptokurtic by size.

### 3. Methods of Estimation of the Parameter of PMMD:

Under this sub-section, the value of the parameter of this distribution has been estimated using the following two methods (a) Method of moments and (b) Maximum likelihood method.

#### (a)Method of moments:

Since there is only one parameter with this distribution, the estimator of the parameter has been obtained using the first moment about the origin of PMMD.

$$\mu_1' = \frac{(6 + 2\alpha + \pi\alpha^2)}{\alpha(2 + \alpha + \pi\alpha^2)}$$

Or, $\mu_1'(2\alpha + \alpha^2 + \pi\alpha^3) - (6 + 2\alpha + \pi\alpha^2) = 0$ (21)

The expression (21) is the polynomial equation of $\alpha$ in third degree and solving this equation by Regula- Falsi method, a point estimator of $\alpha$ may be obtained.

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**(b) Method of maximum likelihood:**

A sample of size n is taken from the PMMD population to get the estimator of the parameter ($\alpha$) from this method as follows

$y_i \; : y_1 \;\; y_2 \;\; y_3 \; \cdots \; y_k$

$f_i \; : f_1 \;\; f_2 \;\; f_3 \; \cdots \; f_k$

The maximum likelihood equation has been obtained as

$$L = \left( \frac{\alpha^3}{2+\alpha+\pi\alpha^2} \right)^n (1+\alpha)^{-\sum_{i=1}^{k}(y_i+3)f_i} \prod_{i=1}^{k} [(1+\alpha)(1+\pi+\pi\alpha+y_i)+(1+y_i)(2+y_i)]^{f_i} \tag{22}$$

Or, $log(L) = 3n\log a - n\log(2+\alpha+\pi\alpha^2) - \left( \sum_{i=1}^{k}(y_i+3) f_i \right)\left(\log(1+\alpha)\right) + \sum_{i=1}^{k} f_i \log\left((1+\alpha)(1+\pi+\pi\alpha+y_i)+(1+y_i)(2+y_i)\right)$

Or, $$\frac{\partial(log(L))}{\partial\alpha} = \frac{3n}{\alpha} - \frac{n(1+2\pi\alpha)}{(2+\alpha+\pi\alpha^2)} - \frac{(\bar{y}n+3n)}{(1+\alpha)} + \sum_{i=1}^{k} \frac{[2\pi(1+\alpha)+(1+\pi+y_i)]f_i}{[(1+\alpha)(1+\pi+\pi\alpha+y_i)+(1+y_i)(2+y_i)]} = 0 \tag{23}$$

An estimate of $\alpha$ can also be obtained by solving the expression (23).

## 4. Goodness of Fit and Applications of PMMD:

In order to check the validity of the theoretical work of this distribution, it seems necessary to describe the application and goodness of fit of this distribution with the support of secondary over-dispersed count data like number of errors per group, number of insects per leaf, number of death due to accident and etcetera used by other researchers. To test goodness of fit the following data are used.

**Example (1): Distribution of mistakes in copying groups of random digits, Kemp and Kemp (1965)**

| **Number of errors per group** | **0** | **1** | **2** | **3** | **$4^+$** |
|---|---|---|---|---|---|
| Observed Frequency | 35 | 11 | 8 | 4 | 2 |

**Example (2): Distribution of Pyrausta nablilalis in 1937, Beall (1940)**

| **Number of insects per leaf** | **0** | **1** | **2** | **3** | **4** | **5** |
|---|---|---|---|---|---|---|
| Observed Frequency | 33 | 12 | 6 | 3 | 1 | 1 |

**Example (3): Distribution of mammalian cytogenic dosimetry lesions in rabbit lymphoblast included by Streptonigrin [NSC-45383], Exposure-70 ($\mu g/kg$)**

| Class / Exposure ($\mu g/kg$) | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| Observed Frequency | 200 | 57 | 30 | 7 | 4 | 0 | 2 |

**Example (4): Distribution of number of red mites on apple leaves, reported by Garman (1923)**

| Number of red mites per leaf | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| Observed Frequency | 38 | 17 | 10 | 9 | 3 | 2 | 1 | |

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## Table I: Observed Verses Expected Frequency of Example (1)

| Number of errors per group | Observed frequency | Expected frequency | | |
|---|---|---|---|---|
| | | PLD | PMD | PMMD |
| 0 | 35 | 33.0 | 32.9 | 33.6 |
| 1 | 11 | 15.3 | 15.3 | 14.7 |
| 2 | 8 | 6.8 | 6.8 | 6.5 |
| 3 | 4 | 2.9 | 3.6 | 2.9 |
| 4 | 2 | 2.0 | 1.4 | 2.3 |
| Total | 60 | 60.0 | 60.0 | 60.0 |
| $\mu'_1$ | 0.78333333 | | | |
| $\mu'_2$ | 1.8500 | | | |
| $\hat{\alpha}$ | | 1.7434 | 2.1654758 | 1.804566256 |
| d.f. | | 2 | 2 | 2 |
| $\chi^2$ | | 1.78 | 1.72 | 1.44 |
| P-value | | 0.61 | 0.625 | 0.677 |

## Table II: Observed Verses Expected Frequency of Example (2)

| Number of insects per leaf | Observed frequency | Expected frequency | | |
|---|---|---|---|---|
| | | PLD | PMD | PMMD |
| 0 | 33 | 31.5 | 31.4 | 32.0 |
| 1 | 12 | 14.2 | 14.2 | 13.7 |
| 2 | 6 | 6.1 | 6.2 | 5.9 |
| 3 | 3 | 2.5 | 2.6 | 2.6 |
| 4 | 1 | 1.0 | 1.0 | 1.1 |
| 5 | 1 | 0.7 | 0.6 | 0.7 |
| Total | 56 | 56.0 | 56.0 | 56.0 |
| $\mu'_1$ | 0.75 | | | |
| $\mu'_2$ | 1.8571 | | | |
| $\hat{\alpha}$ | | 1.8081 | 2.234 | 1.862795711 |
| d.f. | | 2 | 2 | 2 |
| $\chi^2$ | | 0.53 | 0.47 | 0.29 |
| P-value | | 0.83 | 0.85 | 0.89 |

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### Table III: Observed Verses Expected Frequency of Example (3)

| Class per Exposure ($\mu g / kg$) | Observed frequency | Expected frequency | | |
|---|---|---|---|---|
| | | PLD | PMD | PMMD |
| 0 | 200 | 191.8 | 191.6 | 193.4 |
| 1 | 57 | 70.3 | 70.2 | 68.4 |
| 2 | 30 | 24.9 | 25.1 | 24.5 |
| 3 | 7 | 8.6 | 8.7 | 8.8 |
| 4 | 4 | 2.9 | 2.9 | 3.2 |
| 5 | 0 | 1.0 | 1.0 | 1.1 |
| 6 | 2 | 0.5 | 0.5 | 0.6 |
| Total | 300 | 300.0 | 300.0 | 300.0 |
| $\mu'_1$ | 0.553333333 | | | |
| $\mu'_2$ | 1.253333333 | | | |
| $\hat{\alpha}$ | | 2.353339 | 2.784976722 | 2.339317198 |
| d.f. | | 3 | 3 | 3 |
| $\chi^2$ | | 3.91 | 3.81 | 3.04 |
| P-value | | 0.43 | 0.45 | 0.50 |

### Table IV: Observed Verses Expected Frequency of Example (4)

| Number of red mites per leaf | Observed frequency | Expected frequency | |
|---|---|---|---|
| | | PLD | PMMD |
| 0 | 38 | 35.8 | 36.8 |
| 1 | 17 | 20.7 | 20.0 |
| 2 | 10 | 11.4 | 10.9 |
| 3 | 9 | 6.0 | 5.9 |
| 4 | 3 | 3.1 | 3.1 |
| 5 | 2 | 1.6 | 1.6 |
| 6 | 1 | 0.8 | 0.8 |
| $7^+$ | 0 | 0.6 | 0.9 |
| Total | 80 | 80.0 | 80.0 |
| $\mu'_1$ | 1.15 | | |
| $\mu'_2$ | 3.4 | | |
| $\hat{\alpha}$ | | 1.255891 | 1.3724416 |
| d.f. | | 3 | 3 |
| $\chi^2$ | | 2.47 | 1.15 |
| P-value | | 0.61 | 0.82 |

The first example mentioned above related to number of errors per page is due to Kemp and Kemp (Kemp and Kemp, 1965). The second example related to number of insects per leaf is due to Beall (Beall, 1940). The third example related to class per exposer is

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due to Catcheside at al (Catcheside at al, 1946) and the fourth one related to number of red mites per leaf is due to Garman (Garman ,1923). Goodness of fit has been applied in all examples mentioned above and theoretical frequencies extracted using PLD and PMD have been compiled in the table to be compared with the theoretical frequency of PMMD. The first three examples have already been mentioned in the Ph.D. thesis (Sah, 2013).

## CONCLUSION:

In table-V, the degrees of freedom, Chi-square value and P-values of PLD, PMD and PMMD are included to make comparison easy and simple.

**Table V: PLD and PMD Verses PMMD**

| Table | PLD | | | PMD | | | PMMD | | |
|---|---|---|---|---|---|---|---|---|---|
| | $d.f.$ | $\chi^2_{d.f.}$ | $P-Value$ | $d.f.$ | $\chi^2_{d.f.}$ | $P-Value$ | $d.f.$ | $\chi^2_{d.f.}$ | $P-Value$ |
| I | 2 | 1.78 | 0.61 | 2 | 1.72 | 0.625 | 2 | 1.44 | 0.677 |
| II | 2 | 0.53 | 0.83 | 2 | 0.47 | 0.85 | 2 | 0.29 | 0.89 |
| III | 3 | 3.91 | 0.43 | 3 | 3.81 | 0.45 | 3 | 3.04 | 0.50 |
| IV | 3 | 2.47 | 0.61 | - | - | - | 3 | 1.15 | 0.82 |

By using this table and descriptive measures of this distribution, the following conclusion have been drawn.

- In most of the over-dispersed count data sets, it is better alternative of PLD (Sankaran, 1970) and PMD (Sah, 2017) for statistical modeling because P-value obtained by using PMMD is greater than the P-value obtained by using PLD and PMD.
- It is over-dispersed.
- It is positively skewed by shape because $(2/\sqrt{3}) < \gamma_1 < \infty$, and
- It is Leptokurtic by size because $5 < \beta_2 < \infty$.

### Conflict of interest

The authors of this paper have written this paper selflessly with the aim of contributing only to continuous mixtures of Poisson distribution. The authors do not intend to prejudice or offend anyone while writing this paper.

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