Jilin Daxue Xuebao (Gongxueban)/Journal of Jilin University (Engineering and Technology Edition)
ISSN: 1671-5497
E-Publication: Online Open Access
Vol: 43 Issue: 10-2024
DOI: 10.5281/zenodo.13986313

NEW TECHNIQUES TO FIND THE SWIFT CONVERGENCE USING INERTIAL EXTRAPOLATION SCHEME IN THE CAYLEY VARIATIONAL INCLUSION PROBLEM

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Abstract

In this study, we utilize an inertial extrapolation scheme to achieve rapid convergence for the Cayley variational inclusion problem and the equivalent Cayley resolvent equation problem. We have outlined several strategies to address both problems. Still, our primary focus is on validating the rapid convergence for the Cayley variational inclusion problem in a real Banach space and the Cayley resolvent equation problem in a q-uniformly smooth Banach space. We employ an inertial extrapolation strategy in both cases to achieve rapid convergence. A mathematical experiment is presented to demonstrate Swift Convergence.

Keywords: Cayley Approximation Operator, Variational Inclusion, Resolvent Operator, Inertial Extrapolation Scheme, Banach Space, etc.

1. INTRODUCTION

The variational inclusions, developed by Hassouni and Moudafi, are the generalized forms of variational inequalities. Variational inclusions facilitate the examination of a broad spectrum of inter connected and independent problems in the fundamental and applied sciences. Some examples of these issues include those in the domains of elasticity, structural analysis, oceanography, image processing, physics, and engineering sciences. Noor proposed the concept of resolvent equations. The Wiener-Hopf equations are extended and transformed into resolvent equations. Many publications have shown the equality between variational inclusions and resolvent equations. If projection approaches fail to solve the variational inclusion problem, the resolvent operator technique can be employed to solve it. The literature has multiple generalized resolvent operators that involve various monotone operators. Maximal monotone operators are undeniably crucial elements of modern optimization. In addition, the Cayley approximation technique can convert set-valued monotone

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operators into single-valued monotone operators via regularisation. The Cayley approximation operator is applied in various scenarios, such as solving initial value problems for linearised equations of coupled sound and heat flow, describing wave equations as second-order partial differential equations, and modeling heat distribution over time in a fixed region of space using the heat equation. The utilization of generalized resolvent operators has been vital in the advancement of numerous iterative algorithms. However, employing an algorithm that promotes rapid convergence for the sequence generated by the method is consistently beneficial. Various authors have suggested the use of inertial extrapolation systems that incorporate the inertial extrapolation scheme $\{e_n(x_n - x_{n-1})\}$, where e_n represents a factor that enhances the convergence rate of the method. Polyak initially proposed the inertial-type iterative technique for the heavy ball method. The iterative algorithm of inertial nature involves two phases in which the succeeding iterations are derived by utilizing the preceding two terms, as exemplified. In this paper, we analyze the Cayley variational inclusion problem and its corresponding Cayley resolvent equation problem, which was described earlier. In addition, we explore various approaches to address the Cayley resolvent equation problem and Cayley variational inclusion. Our research is centered on the speedy convergence of both issues through the utilization of an inertial extrapolation scheme.

2. FUNDAMENTAL TOOLS AND CONCEPTS

Let us consider $\widetilde{\mathcal{B}}$ is called a real Banach Space and $\mathcal{B}^*$ is its topological dual equipped with the norm $\| \, . \, \|$ and duality pairing $\langle \, .\, ,. \, \rangle$ between $\widetilde{\mathcal{B}}$ and $\mathcal{B}^*$. Consider $2^{\widetilde{\mathcal{B}}}$ represents the set of all non-empty subsets of $\widetilde{\mathcal{B}}$ and $C(\widetilde{\mathcal{B}})$ be the family of nonempty compact subsets of $\widetilde{\mathcal{B}}$

For $q > 1$, the generalized duality mapping $\mathcal{N}_q : \widetilde{\mathcal{B}} \rightarrow \mathcal{B}^*$ is defined by $\mathcal{N}_q(x) = \{y \in \mathcal{B}^* : \langle \, x, y \rangle = \|x\|^q \text{ and } \| \, y \, \| = \|x\|^{q-1}\} \quad \forall, x \in \widetilde{\mathcal{B}}.$

If $q = 2$ then $\mathcal{N}_q$ is called normalized duality mapping. Especially, $\mathcal{N} := \mathcal{N}_2$ become be a normalized duality mapping on $\widetilde{\mathcal{B}}$.Then very familiar that $\mathcal{N}_q(x) = \|x\|^{q-2}\mathcal{N}_2(x)$ when $x \neq 0$ and $\mathcal{N}_q(x)$ be a subdifferential of functional $(\frac{1}{q})\| \, . \, \|^q$ at x. The mapping $\mathcal{N}_q$ is single-valued if $\widetilde{\mathcal{B}}$ is uniformly smooth.

Lemma 1. A real uniformly smooth Banach space $\widetilde{\mathcal{B}}$ is q-uniformly smooth if there exists a

constant $C_q > 0$ such that $\|x + y\|^q \leq \|x\|^q + q\langle \, y, \mathcal{N}_q(x) \rangle + C_q\|y\|^q \quad \forall, x, y \in \widetilde{\mathcal{B}}$

We mention the following standard definitions before offering those necessary for the paper's presentation and readers' convenience. As a result, let us consider a real Hilbert space $\widetilde{\mathcal{B}} = H$.

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Definition 2.1. A single-valued mapping $\tilde{S} : H \to H$ is said to be

(i) Monotone if

$$\langle \tilde{S}(x) - \tilde{S}(y),\ x - y \rangle \geq 0 \qquad \forall, x, y \in H$$

(ii) Strongly monotone if there exists constant $\delta_{\tilde{S}} \geq 0$ such that

$$\langle \tilde{S}(x) - \tilde{S}(y), x - y \rangle \geq \delta_{\tilde{S}} \|x + y\|^2 \quad \forall, x, y \in H$$

Definition 2.2. A set-valued mapping $\mathfrak{D} : H \to 2^H$ is said to be monotone for all $u \in \mathfrak{D}(x)$,

$v \in \mathfrak{D}(y)$ if

$\langle u - v, x - y \rangle \geq 0 \quad \forall, x, y \in H,$

Definition 2.3. Let us consider $\tilde{S} : H \to H$ be a single-valued mapping. A set-valued mapping $\mathfrak{D} : H \to 2^H$ is said to be $\tilde{S}$-monotone if $\mathfrak{D}$ is monotone and

$[\tilde{S} + \gamma \mathfrak{D}]H = H, \quad \gamma > 0$ is a constant

This paper's presentation requires the following generalizations of Definitions 2.1-2.3 above in a *q*-uniformly smooth Banach space.

Definition 2.4. A single-valued mapping $\tilde{S} : \widetilde{\mathcal{B}} \to \widetilde{\mathcal{B}}$ is said to be

(i) Accretive if

$$\langle \tilde{S}(x) - \tilde{S}(y), \mathcal{N}_q(x - y) \rangle \geq 0 \qquad \forall, x, y \in \widetilde{\mathcal{B}}$$

(ii) Strongly accretive if there exists constant $\delta_{\tilde{S}} \geq 0$ such that

$$\langle \tilde{S}(x) - \tilde{S}(y), \mathcal{N}_q(x - y \rangle \geq \delta_{\tilde{S}} \|x - y\|^q \qquad \forall x, y \in \widetilde{\mathcal{B}}$$

(iii) Lipschitz continuous if there exists constant $\lambda_{\tilde{S}} \geq 0$ such that

$$\|\tilde{S}(x) - \tilde{S}(y)\| \leq \lambda_{\tilde{S}} \|x - y\|, \quad \forall, x, y \in \widetilde{\mathcal{B}}$$

Definition 2.5. A set-valued mapping $\mathfrak{D} : \widetilde{\mathcal{B}} \to 2^{\widetilde{\mathcal{B}}}$ is said to be accretive for all $u \in \mathfrak{D}(x)$,

$v \in \mathfrak{D}(y)$ if

$\langle u - v, \mathcal{N}_q(x - y) \rangle \geq 0 \quad \forall\, x, y \in H,$

Definition 2.6. Consider $\tilde{S} : \widetilde{\mathcal{B}} \to \widetilde{\mathcal{B}}$ be a mapping. The set-valued mapping $\mathfrak{D} : \widetilde{\mathcal{B}} \to 2^{\widetilde{\mathcal{B}}}$ is said to be $\tilde{S}$ accretive if $\mathfrak{D}$ is accretive and

$[\tilde{S} + \gamma \mathfrak{D}]\widetilde{\mathcal{B}} = \widetilde{\mathcal{B}} \quad \gamma > 0$ is a constant.

It is commonly recognized that the resolvent operator of the type $[I + \gamma \mathfrak{D}]^{-1}$ where $\mathfrak{D}$ is a set-valued monotone mapping, γ is a positive constant, and I is the identity mapping, providing the foundation for all splitting techniques (2).

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Definition 2.7. The resolvent operator $\mathfrak{R}_{I,\gamma}^{\mathfrak{D}} : H \to H$ which is denoted by

$$\mathfrak{R}_{I,\gamma}^{\mathfrak{D}}(x) = [I + \gamma\mathfrak{D}]^{-1}(x) \quad \forall, x \in H,$$

I is identity mapping and $\gamma > 0$ is a constant.

Definition 2.8. The Cayley approximation operator $C_{I,\gamma}^{\mathfrak{D}} : H \to H$ which is denoted by

$$C_{I,\gamma}^{\mathfrak{D}}(x) = [2\mathfrak{R}_{I,\gamma}^{\mathfrak{D}} - I](x) \qquad \forall, x \in H$$

I is identity mapping and $\gamma > 0$ is a constant.

Definition 2.9. Let us consider $\tilde{S} : \widetilde{\mathcal{B}} \to \widetilde{\mathcal{B}}$ is a single-valued mapping and $\mathfrak{D} : \widetilde{\mathcal{B}} \to 2^{\widetilde{\mathcal{B}}}$ is a set-valued mapping. The generalized resolvent operator $\mathfrak{R}_{\tilde{S},\gamma}^{\mathfrak{D}} : \widetilde{\mathcal{B}} \to \widetilde{\mathcal{B}}$ with respect to $\tilde{S}$ and $\mathfrak{D}$ which is denoted by

$$\mathfrak{R}_{\tilde{S},\gamma}^{\mathfrak{D}}(x) = \left[\tilde{S} + \gamma\mathfrak{D}\right]^{-1}(x) \qquad \forall, x \in H$$

$\gamma > 0$ is a constant.

Definition 2.10. The generalized Cayley approximation operator $C_{\tilde{S},\gamma}^{\mathfrak{D}} : \widetilde{\mathcal{B}} \to \widetilde{\mathcal{B}}$ which is denoted by

$$C_{\tilde{S},\gamma}^{\mathfrak{D}}(x) = \left[2\mathfrak{R}_{\tilde{S},\gamma}^{\mathfrak{D}} - \tilde{S}\right](x) \qquad \forall, x \in H$$

$\gamma > 0$ is a constant.

Proposition 1. Let us consider $\tilde{S} : \widetilde{\mathcal{B}} \to \widetilde{\mathcal{B}}$ is called strongly accretive mapping with constant r and $\mathfrak{D} : \widetilde{\mathcal{B}} \to 2^{\widetilde{\mathcal{B}}}$ is $\tilde{S}$ accretive set-valued mapping. Then the generalized resolvent operator $\mathfrak{R}_{\tilde{S},\gamma}^{\mathfrak{D}} : \widetilde{\mathcal{B}} \to \widetilde{\mathcal{B}}$ is Lipschitz continuous with constant $\frac{1}{r}$ such that

$$\left\|\mathfrak{R}_{\tilde{S},\gamma}^{\mathfrak{D}}(x) - \mathfrak{R}_{\tilde{S},\gamma}^{\mathfrak{D}}(y)\right\| \le \frac{1}{r}\|x - y\| \qquad \forall, x, y \in \widetilde{\mathcal{B}}$$

Definition 2.11. A single-valued mapping $A : \widetilde{\mathcal{B}} \to \widetilde{\mathcal{B}}$ is called Lipschitz continuous if there exists a constant $\lambda_A \ge 0$ such that

$$\|A(x) - A(y)\| \ge \lambda_A\|x - y\| \quad \forall, x, y \in \widetilde{\mathcal{B}}$$

Definition 2.12. Consider a multi-valued mapping $\tilde{S} : \widetilde{\mathcal{B}} \to \widetilde{\mathcal{B}}$ is said to be D-Lipschitz continuous. Then there exists a constant $\lambda_{D_{\tilde{S}}} > 0$ such that

$$D\left(\tilde{S}(x), \tilde{S}(y)\right) \le \lambda_{D_{\tilde{S}}}\|x - y\| \quad \forall, x, y \in C(\widetilde{\mathcal{B}})$$

Definition 2.13. Suppose $N : H \times H \to H$ is a single-valued mapping and $\mathfrak{D} : \widetilde{\mathcal{B}} \times \widetilde{\mathcal{B}} \to 2^{\widetilde{\mathcal{B}}}$ is a multi-valued mapping. Then

(i) N is said to be Lipschitz continuous in the first argument if there exists a constant $\lambda_{N_1} > 0$ for all $x, y \in H, u_1 \in \mathfrak{D}(x), u_2 \in \mathfrak{D}(y)$ such that

$$\|N(u_1, .) - N(u_2, .)\| \le \lambda_{N_1}\|u_1 - u_2\|$$

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(ii) N is said to be Lipschitz continuous in the first argument if there exists a constant $\lambda_{N_2} > 0$, for all $x, y \in H$, $v_1 \in \mathfrak{D}(x), v_2 \in \mathfrak{D}(y)$ such that

$$\|N(.,v_1) - N(.,v_2)\| \le \lambda_{N_2}\|v_1 - v_2\|$$

Lemma 2. Consider $\{l_n\}$ be a sequence of non-negative real numbers such that

$$S_{n+1} \le (1-\beta_n)l_n + \beta_n\sigma_n + \widetilde{\gamma}_n \qquad \forall, n > 1$$

Where, 1. $\{\beta_n\} \subset [0,1]$, $\sum_{n=1}^{\infty}\beta_n = \infty$

2. *lim sup*$\sigma_n \le 0$

3. $\widetilde{\gamma}_n \ge 0$, $(n \ge 1)$, $\sum_{n=1}^{\infty}\beta_n < \infty$

Then $l_n \to 0$, as $n \to \infty$

Proposition 2. (i) If $\tilde{S} : \widetilde{\mathcal{B}} \to \widetilde{\mathcal{B}}$ is r-strongly accretive, $\beta_{\tilde{S}}$ - expansive, $\lambda_{\tilde{S}}$ -Lipschitz continuous and generalized resolvent operator $\mathfrak{R}_{\tilde{S},\gamma}^{\mathfrak{D}} : \widetilde{\mathcal{B}} \to \widetilde{\mathcal{B}}$ is $\frac{1}{r}$-Lipschitz continuous and the generalized Cayley approximation operator $C_{\tilde{S},\gamma}^{\mathfrak{D}} : \widetilde{\mathcal{B}} \to \widetilde{\mathcal{B}}$ associated with $\tilde{S}$.Then the generalized Cayley approximation operator $C_{\tilde{S},\gamma}^{\mathfrak{D}} : \widetilde{\mathcal{B}} \to \widetilde{\mathcal{B}}$ is θ_c-strongly accretive associate with $\tilde{S}$, then we have

$$\langle C_{\tilde{S},\gamma}^{\mathfrak{D}}(x) - C_{\tilde{S},\gamma}^{\mathfrak{D}}(y), \mathcal{N}_q(\tilde{S}(x) - \tilde{S}(y)\rangle \ge \theta_c\|x-y\|^q$$

Where $\theta_c = \frac{2\lambda_{\tilde{S}}^{q-1} - r\beta_{\tilde{S}}^q}{r}$, $\gamma r \neq 0$, $\beta_{\tilde{S}}^q r > \lambda_c\lambda_{\tilde{S}}^{q-1}$ $\qquad \forall, \gamma, \beta, \lambda > 0$

(ii) If $\tilde{S}$ is $\lambda_{\tilde{S}}$ -Lipschitz continuous, *r*-strongly accretive and $\mathfrak{R}_{\tilde{S},\gamma}^{\mathfrak{D}}$ is $\frac{1}{r}$ -Lipschitz continuous, then the generalized Cayley approximation operator $C_{\tilde{S},\gamma}^{\mathfrak{D}}$ is λ_C -lipschitz continuous that is

$$\left\|C_{\tilde{S},\gamma}^{\mathfrak{D}}(x) - C_{\tilde{S},\gamma}^{\mathfrak{D}}(y)\right\| \le \lambda_C\|x-y\| \text{ , where } \lambda_C = \frac{\lambda_{\tilde{S}}r+2}{r}$$

Proof: (i) From the definition of generalized duality mapping, expansiveness and Lipschitz continuity of $\mathfrak{R}_{\tilde{S},\gamma}^{\mathfrak{D}}$, $C_{\tilde{S},\gamma}^{\mathfrak{D}}$ and $\tilde{S}$, We have,

$$\begin{aligned}
&\langle C_{\tilde{S},\gamma}^{\mathfrak{D}}(x) - C_{\tilde{S},\gamma}^{\mathfrak{D}}(y), \mathcal{N}_q(\tilde{S}(x) - \tilde{S}(y)\rangle \\
&= \langle \left(2\mathfrak{R}_{\tilde{S},\gamma}^{\mathfrak{D}} - \tilde{S}\right)(x) - \left(2\mathfrak{R}_{\tilde{S},\gamma}^{\mathfrak{D}} - \tilde{S}\right)(y), \mathcal{N}_q(\tilde{S}(x) - \tilde{S}(y)\rangle \\
&= \langle 2\mathfrak{R}_{\tilde{S},\gamma}^{\mathfrak{D}}(x) - \tilde{S}(x) - 2\mathfrak{R}_{\tilde{S},\gamma}^{\mathfrak{D}}(y) + \tilde{S}(y), \mathcal{N}_q(\tilde{S}(x) - \tilde{S}(y)\rangle \\
&= \langle 2(\mathfrak{R}_{\tilde{S},\gamma}^{\mathfrak{D}}(x) - \mathfrak{R}_{\tilde{S},\gamma}^{\mathfrak{D}}(y)) - (\tilde{S}(x) - \tilde{S}(y)), \mathcal{N}_q(\tilde{S}(x) - \tilde{S}(y)\rangle \\
&= 2\langle(\mathfrak{R}_{\tilde{S},\gamma}^{\mathfrak{D}}(x) - \mathfrak{R}_{\tilde{S},\gamma}^{\mathfrak{D}}(y)), \mathcal{N}_q(\tilde{S}(x) - \tilde{S}(y)\rangle \\
&\quad - \langle(\tilde{S}(x) - \tilde{S}(y)), \mathcal{N}_q(\tilde{S}(x) - \tilde{S}(y)\rangle
\end{aligned}$$

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$$\geq 2\| \mathfrak{R}^{\mathfrak{D}}_{\tilde{S},\gamma}(x) - \mathfrak{R}^{\mathfrak{D}}_{\tilde{S},\gamma}(y)\| \|\tilde{S}(x) - \tilde{S}(y)\|^{q-1} - \|\tilde{S}(x) - \tilde{S}(y)\|^{q}$$
$$\geq \frac{2}{r}\|x-y\|\lambda_{\tilde{S}}^{q-1}\|x-y\|^{q-1} - \beta_{\tilde{S}}^{q}\|x-y\|^{q}$$
$$\geq \frac{2}{r}\lambda_{\tilde{S}}^{q-1}\|x-y\|^{q} - \beta_{\tilde{S}}^{q}\|x-y\|^{q}$$
$$\geq (\frac{2}{r}\lambda_{\tilde{S}}^{q-1} - \beta_{\tilde{S}}^{q})\|x-y\|^{q}$$
$$\geq \frac{1}{r}(2\lambda_{\tilde{S}}^{q-1} - r\beta_{\tilde{S}}^{q})\|x-y\|^{q}$$
$$\geq \theta_c\|x-y\|^{q} \qquad \text{Where } \theta_c = \frac{1}{r}(2\lambda_{\tilde{S}}^{q-1} - r\beta_{\tilde{S}}^{q}),$$

Thus, the generalized Cayley approximation operator is θ_c - strongly accretive concerning $\tilde{S}$.

(i) Using Lipschitz continuity of accretive $\tilde{S}$ and generalized resolvent operator $R^{\mathfrak{D}}_{\tilde{S},\gamma}$

We evaluate $\|C^{\mathfrak{D}}_{\tilde{S},\gamma}(x) - C^{\mathfrak{D}}_{\tilde{S},\gamma}(y)\|$

$$= \|(2\mathfrak{R}^{\mathfrak{D}}_{\tilde{S},\gamma}(x) - \tilde{S}(x)) - (2\mathfrak{R}^{\mathfrak{D}}_{\tilde{S},\gamma}(y) - \tilde{S}(y))\|$$
$$= 2\|\mathfrak{R}^{\mathfrak{D}}_{\tilde{S},\gamma}(x) - \mathfrak{R}^{\mathfrak{D}}_{S,\gamma}(y)\| - \|\tilde{S}(x) - \tilde{S}(y)\|$$
$$\leq 2\frac{1}{r}\|x-y\| + \lambda_{\tilde{S}}\|x-y\|$$
$$\leq (\frac{2}{r} + \lambda_{\tilde{S}})\|x-y\|$$
$$\leq \lambda_C\|x-y\| \quad \text{where, } \lambda_c = \frac{1}{r}(2 + r\lambda_{\tilde{S}})$$

Thus, the generalized Cayley approximation operator $C^{\mathfrak{D}}_{\tilde{S},\gamma}$ is λ_C –Lipschitz continuous.

3. STATEMENT OF THE CAYLEY INCLUSION PROBLEM

Let $\tilde{S} : \widetilde{\mathcal{B}} \to \widetilde{\mathcal{B}}$ be a single-valued mapping and also $N : \widetilde{\mathcal{B}} \times \widetilde{\mathcal{B}} \to \widetilde{\mathcal{B}}$ be another single-valued mapping and $\mathfrak{D} : \widetilde{\mathcal{B}} \to 2^{\widetilde{\mathcal{B}}}$ be a set-valued mapping and $\tilde{P}, \tilde{Q} : \widetilde{\mathcal{B}} \to C(\widetilde{\mathcal{B}})$ are multi-valued mapping. Let $C^{\mathfrak{D}}_{\tilde{S},\gamma}$ be the generalized the Cayley approximation operator. We consider the following Cayley variational inclusions problem.

Find $x \in \widetilde{\mathcal{B}}$, $u \in \tilde{P}(x), v \in \tilde{Q}(x)$ such that

$$0 \in C^{\mathfrak{D}}_{\tilde{S},\gamma}(x) + N(u,v) + \mathfrak{D}(x) \qquad (1)$$

If $C^{\mathfrak{D}}_{\tilde{S},\gamma}(x) = 0$ and $N(u,v) = 0$ then the problem (1) reduces to the problem of finding $x \in \widetilde{\mathcal{B}}$ such that

$0 \in \mathfrak{D}(x)$, which is the fundamental problem represented by Rockafellar.

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E-Publication: Online Open Access
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Lemma 3. Cayley variational inclusion problem (1) has the solutions $x \in \widetilde{\mathcal{B}}$, $u \in P(x)$, $v \in Q(x)$ if and only if the following equation is satisfied:

$$x = \mathfrak{R}^{\mathfrak{D}}_{\tilde{S},\gamma}[\tilde{S}(x) - \gamma N(u,v) - \gamma C^{\mathfrak{D}}_{\tilde{S},\gamma}(x)] \quad (2)$$

Proof: Let, $x \in \widetilde{\mathcal{B}}$ satisfies the equation (2).

$$\text{Then } x = \mathfrak{R}^{\mathfrak{D}}_{\tilde{S},\gamma}[\tilde{S}(x) - \gamma N(u,v) - \gamma C^{\mathfrak{D}}_{\tilde{S},\gamma}(x)]$$

$$Or, x = \left[\tilde{S} + \gamma\mathfrak{D}\right]^{-1}[\tilde{S}(x) - \gamma N(u,v) - \gamma C^{\mathfrak{D}}_{\tilde{S},\gamma}(x)]$$

$$Or, \left[\tilde{S} + \gamma\mathfrak{D}\right](x) = [S(x) - \gamma N(u,v) - \gamma C^{\mathfrak{D}}_{\tilde{S},\gamma}(x)]$$

$$Or, \tilde{S}(x) + \gamma\mathfrak{D}(x) = S(x) - \gamma N(u,v) - \gamma C^{\mathfrak{D}}_{\tilde{S},\gamma}(x)$$

$$Or,\ \gamma\mathfrak{D}(x) = -\gamma N(u,v) - \gamma C^{\mathfrak{D}}_{\tilde{S},\gamma}(x)$$

$0 \in C^{\mathfrak{D}}_{\tilde{S},\gamma}(x) + N(u,v) + \mathfrak{D}(x)$ which is the required Cayley variational inclusion problem (1).

Iterative Algorithm 1: Determine the sequence $\{x_n\}$,$\{u_n\}$, and $\{v_n\}$ for any $x_0 \in \widetilde{\mathcal{B}}$ $u_0 \in \tilde{P}(x_0)$, and $v_0 \in \tilde{Q}(x_0)$, from the following Scheme

$$x_{n+1} = \mathfrak{R}^{\mathfrak{D}}_{\tilde{S},\gamma}[\tilde{S}(x_n) - \gamma N(u_n,v_n) - \gamma C^{\mathfrak{D}}_{\tilde{S},\gamma}(x_n)] \quad (3)$$

where, $n = 0,1,2,3, \dots \dots \dots$ and $\gamma > 0$ is a constant.

Another way to write equation (2) is as follows:

$$x = \mathfrak{R}^{\mathfrak{D}}_{\tilde{S},\gamma}[\frac{\tilde{S}(x)+\tilde{S}(x)}{2} - \gamma N(u,v) - \gamma C^{\mathfrak{D}}_{\tilde{S},\gamma}(x)] \quad (4)$$

We propose the following iterative strategy based on (4).

Iterative Algorithm 2: Determine x_{n+1}, u_{n+1} and v_{n+1} using the recurrence relation for any $x_0 \in \widetilde{\mathcal{B}}$, $u_0 \in \tilde{P}(x_0)$, and $v_0 \in \tilde{Q}(x_0)$, we have

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n \mathfrak{R}^{\mathfrak{D}}_{\tilde{S},\gamma}[\frac{\tilde{S}(x_n)+\tilde{S}(x_{n+1})}{2} - \gamma N(u_{n+1},v_{n+1}) - \gamma C^{\mathfrak{D}}_{\tilde{S},\gamma}(x_{n+1})] \quad (5)$$

where, $n = 0,1,2,3, \dots \dots \dots$ and $\gamma > 0$ is a constant and $\alpha_n \in [0,1]$

The predictor-corrector approach is used to describe the following inertial extrapolation scheme.

Iterative Algorithm 3: Determine x_{n+1}, u_{n+1} and v_{n+1} using the recurrence relation for any $x_0 \in \widetilde{\mathcal{B}}$, $u_0 \in \tilde{P}(x_0)$, and $v_0 \in \tilde{Q}(x_0)$, we have

$$w_n = x_n + e_n(x_n - x_{n-1}) \quad (6)$$

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n \mathfrak{R}^{\mathfrak{D}}_{\tilde{S},\gamma}[\frac{\tilde{S}(x_n)+\tilde{S}(w_n)}{2} - \gamma N(u_n,v_n) - \gamma C^{\mathfrak{D}}_{\tilde{S},\gamma}(w_n)] \quad (7)$$

where, $\gamma > 0$ is a constant and $e_n, \alpha_n \in [0,1]$, e_n is the extrapolating term. $\forall, n \geq 1$.

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4. CAYLEY RESOLVENT INCLUSION PROBLEM.

To get the existence and convergence result for the Cayley variational inclusion problem (1), one can utilize the aforementioned algorithms 1 and 2. Finally, we provide a convergence result for the Cayley variational inclusion problem (1) in the sequel by using the inertial extrapolation scheme 3. Regarding the Cayley variational inclusion problem (1), we formulate the following Cayley resolvent equation problem.

Find $x, \tilde{z} \in \widetilde{\mathcal{B}}$, $u \in \tilde{P}(x), v \in \tilde{Q}(x)$ such that

$$C_{\tilde{S},\gamma}^{\mathfrak{D}}(x) + N(u,v) + \gamma^{-1} T_{\tilde{S},\gamma}^{\mathfrak{D}}(\tilde{z}) = 0 \tag{8}$$

Where, $T_{\tilde{S},\gamma}^{\mathfrak{D}}(\tilde{z}) = \left[I - \tilde{S}R_{\tilde{S},\gamma}^{\mathfrak{D}}\right](\tilde{z})$ and $\tilde{S}[R_{\tilde{S},\gamma}^{\mathfrak{D}}(\tilde{z})] = \left[\tilde{S}(R_{\tilde{S},\gamma}^{\mathfrak{D}})\right](\tilde{z})$ (A)

The following Lemma ensures that the Cayley variational inclusion issue (1) and the Cayley resolvent equation problem (8) are comparable.

Lemma 4. The Cayley variational inclusion problem (1) has the solutions $x \in \widetilde{\mathcal{B}}$, $u \in \tilde{P}(x)$,

$v \in \tilde{Q}(x)$ if and only if the Cayley resolvent equation problem (8) has the solutions $x, \tilde{z} \in \widetilde{\mathcal{B}}$, $u \in \tilde{P}(x), v \in \tilde{Q}(x)$ that is, $\tilde{s}$ is one-one and

$$x = R_{\tilde{S},\gamma}^{\mathfrak{D}}(\tilde{z}) \tag{9}$$

$$\tilde{z} = \tilde{S}(x) - \gamma N(u,v) - \gamma C_{\tilde{S},\gamma}^{\mathfrak{D}}(x) \tag{10}$$

where, $\gamma > 0$ is a constant.

Proof: consider $x \in \widetilde{\mathcal{B}}$, $u \in \tilde{P}(x)$, and $v \in \tilde{Q}(x)$ are the solutions to Cayley's variational inclusion problem (1). Then according to Lemma 3, it fulfills the formula:

$$x = \Re_{\tilde{S},\gamma}^{\mathfrak{D}}[\tilde{S}(x) - \gamma N(u,v) - \gamma C_{\tilde{S},\gamma}^{\mathfrak{D}}(x)]$$

Where, $x = R_{\tilde{S},\gamma}^{\mathfrak{D}}(\tilde{z})$

and $\tilde{z} = \tilde{S}(x) - \gamma N(u,v) - \gamma C_{\tilde{S},\gamma}^{\mathfrak{D}}(x)$

Using (9), (10) becomes

$$\tilde{z} = \tilde{S}(\Re_{\tilde{S},\gamma}^{\mathfrak{D}}(\tilde{z})) - \gamma N(u,v) - \gamma C_{\tilde{S},\gamma}^{\mathfrak{D}}(x)$$

$$Or, [I - \tilde{S}(R_{\tilde{S},\gamma}^{\mathfrak{D}})](\tilde{z}) = -\gamma N(u,v) - \gamma C_{\tilde{S},\gamma}^{\mathfrak{D}}(x)$$

$$Or, T_{\tilde{S},\gamma}^{\mathfrak{D}}(\tilde{z}) = -\gamma N(u,v) - \gamma C_{\tilde{S},\gamma}^{\mathfrak{D}}(x) \quad \text{[By (A)]}$$

$$Or, -\gamma^{-1} T_{\tilde{S},\gamma}^{\mathfrak{D}}(\tilde{z}) = -N(u,v) - C_{\tilde{S},\gamma}^{\mathfrak{D}}(x)$$

Thus $C_{\tilde{S},\gamma}^{\mathfrak{D}}(x) + N(u,v) + \gamma^{-1} T_{\tilde{S},\gamma}^{\mathfrak{D}}(\tilde{z}) = 0$ which is the required Cayley resolvent equation problem (8).

Jilin Daxue Xuebao (Gongxueban)/Journal of Jilin University (Engineering and Technology Edition)
ISSN: 1671-5497
E-Publication: Online Open Access
Vol: 43 Issue: 10-2024
DOI: 10.5281/zenodo.13986313

Conversely, Let, $x, \tilde{z} \in \widetilde{\mathcal{B}}$ $u \in \tilde{P}(x)$, and $v \in \tilde{Q}(x)$ are the solutions to the Cayley resolvent equation problem (8).

Then we have, $C^{\mathfrak{D}}_{\tilde{S},\gamma}(x) + N(u,v) + \gamma^{-1} T^{\mathfrak{D}}_{\tilde{S},\gamma}(\tilde{z}) = 0$

$$Or, -\gamma^{-1} T^{\mathfrak{D}}_{\tilde{S},\gamma}(\tilde{z}) = N(u,v) + C^{\mathfrak{D}}_{\tilde{S},\gamma}(x)$$

$$Or, C^{\mathfrak{D}}_{\tilde{S},\gamma}(x) + N(u,v) = -\gamma^{-1}[I - \tilde{S}(R^{\mathfrak{D}}_{\tilde{S},\gamma})](\tilde{z})$$

$$Or, \gamma C^{\mathfrak{D}}_{\tilde{S},\gamma}(x) + \gamma N(u,v) = [\tilde{S}(\mathfrak{R}^{\mathfrak{D}}_{\tilde{S},\gamma})](\tilde{z}) - (\tilde{z})$$

$$Or, \gamma C^{\mathfrak{D}}_{\tilde{S},\gamma}(x) + \gamma N(u,v) = [\tilde{S}(\mathfrak{R}^{\mathfrak{D}}_{\tilde{S},\gamma})][\tilde{S}(x) - \gamma N(u,v) - \gamma C^{\mathfrak{D}}_{\tilde{S},\gamma}(x)]$$
$$-(\tilde{S}(x) - \gamma N(u,v) - \gamma C^{\mathfrak{D}}_{\tilde{S},\gamma}(x))$$

This implies that, $\tilde{S}(x) = \tilde{S}[\mathfrak{R}^{\mathfrak{D}}_{\tilde{S},\gamma}(\tilde{S}(x) - \gamma N(u,v) - \gamma C^{\mathfrak{D}}_{\tilde{S},\gamma}(x)]$

Since $\tilde{S}$ is one – one, we have, $x = \mathfrak{R}^{\mathfrak{D}}_{\tilde{S},\gamma}[\tilde{S}(x) - \gamma N(u,v) - \gamma(C^{\mathfrak{D}}_{\tilde{S},\gamma}(x))]$

The solutions to the Cayley variational inclusion problem (1) are $x \in \widetilde{\mathcal{B}}, u \in \tilde{P}(x)$, and $v \in \tilde{Q}(x)$ as indicated by Lemma 3. We present the Cayley resolvent equation problem (8) solution strategy based on Lemma 4.

Iterative Algorithm 4: Utilizing the following Scheme, determine the sequence $\{x_n\}$, $\{z_n\}$, $\{u_n\}$, and $\{v_n\}$ for every $x_0, \hat{z}_0 \in \widetilde{\mathcal{B}}$, $u_0 \in \tilde{P}(x_0)$, and $v_0 \in \tilde{Q}(x_0)$,

$$x_n = \mathfrak{R}^{\mathfrak{D}}_{\tilde{S},\gamma}(\hat{z}_n) \tag{11}$$

$$\hat{z}_{n+1} = \tilde{S}(x_n) - N(u_n, v_n) - \gamma(C^{\mathfrak{D}}_{\tilde{S},\gamma}(x_n)) \tag{12}$$

where, n = 0,1,2, 3…….and $\gamma > 0$ is a constant

Now we rewrite the Cayley resolvent equation problem (8)

$$\hat{z} = \tilde{S}(x) - N(u,v) - C^{\mathfrak{D}}_{\tilde{S},\gamma}(x) + (I - \gamma^{-1})\, T^{\mathfrak{D}}_{\tilde{S},\gamma}(\tilde{z}) \tag{13}$$

Verification.

We have from (9).

$$\hat{z} = \tilde{S}\left(\mathfrak{R}^{\mathfrak{D}}_{\tilde{S},\gamma}(\tilde{z})\right) - N(u,v) - C^{\mathfrak{D}}_{\tilde{S},\gamma}(x) + T^{\mathfrak{D}}_{\tilde{S},\gamma}(\tilde{z}) - \gamma^{-1} T^{\mathfrak{D}}_{\tilde{S},\gamma}(\tilde{z})$$

Since $[I - \tilde{S}\,\mathfrak{R}^{\mathfrak{D}}_{\tilde{S},\gamma}] = T^{\mathfrak{D}}_{\tilde{S},\gamma}$ Then we have,

$$[I - \tilde{S}\mathfrak{R}^{\mathfrak{D}}_{\tilde{S},\gamma}](\tilde{z}) = -N(u,v) - C^{\mathfrak{D}}_{\tilde{S},\gamma}(x) + T^{\mathfrak{D}}_{\tilde{\gamma},\xi}(\tilde{z}) - \gamma^{-1} T^{\mathfrak{D}}_{\tilde{S},\gamma}(\tilde{z})$$

$$Or, T^{\mathfrak{D}}_{\tilde{S},\gamma}(\tilde{z}) = -N(u,v) - C^{\mathfrak{D}}_{\tilde{S},\gamma}(x) + T^{\mathfrak{D}}_{\tilde{S},\gamma}(\tilde{z}) - \gamma^{-1} T^{\mathfrak{D}}_{\tilde{S},\gamma}(\tilde{z})$$

$$Or, C^{\mathfrak{D}}_{\tilde{S},\gamma}(x) + N(u,v) + \gamma^{-1} T^{\mathfrak{D}}_{\tilde{S},\gamma}(\tilde{z}) = 0$$

We propose the iteration approach below, based on the fixed-point formulation (13)

ISSN: 1671-5497
E-Publication: Online Open Access
Vol: 43 Issue: 10-2024
DOI: 10.5281/zenodo.13986313

Iterative Algorithm 5. Utilizing the following Scheme, determine the sequence $\{x_n\}$, $\{z_n\}$, $\{u_n\}$, and $\{v_n\}$ for every $x_0, \hat{z}_0 \in \widetilde{\mathcal{B}}, u_0 \in \tilde{P}(x_0)$, and $v_0 \in \tilde{Q}(x_0)$,

$$x_n = \Re^{\mathfrak{D}}_{\tilde{S},\gamma}(\hat{z}_n)$$

and $$\hat{z}_{n+1} = \tilde{S}(x_n) - N(u_n, v_n) - C^{\mathfrak{D}}_{\tilde{S},\gamma}(x) + (I - \gamma^{-1})\, T^{\mathfrak{D}}_{\tilde{S},\gamma}(\hat{z}_n)$$

where, $n = 0,1,2,3, \dots\dots\dots$ and $\gamma, \delta > 0$ is a constant

The Cayley resolvent equation problem (8) can also be expressed as follows:

$$x = x - \delta[\tilde{z} - \tilde{S}\left(\Re^{\mathfrak{D}}_{\tilde{S},\gamma}(\tilde{z})\right) + \gamma N(u,v) + \gamma C^{\mathfrak{D}}_{\tilde{S},\gamma}(x)] \quad \forall, \delta \geq 1 \quad (14)$$

Verification:

$$x = x - \delta[[I - \tilde{S}(\Re^{\mathfrak{D}}_{\tilde{S},\gamma})](\tilde{z}) + \gamma N(u,v) + \gamma C^{\mathfrak{D}}_{\tilde{S},\gamma}(x)]$$

$$Or, x = x - \delta[T^{\mathfrak{D}}_{\tilde{S},\gamma}(\tilde{z}) + \gamma N(u,v) + \gamma C^{\mathfrak{D}}_{\tilde{S},\gamma}(x)]$$

$$Or, \delta\left[T^{\mathfrak{D}}_{\tilde{S},\gamma}(\tilde{z}) + \gamma N(u,v) + \gamma C^{\mathfrak{D}}_{\tilde{S},\gamma}(x)\right] = 0$$

$$Or, T^{\mathfrak{D}}_{\tilde{S},\gamma}(\tilde{z}) + \gamma N(u,v) + \gamma C^{\mathfrak{D}}_{\tilde{S},\gamma}(x) = 0$$

$Or, C^{\mathfrak{D}}_{\tilde{S},\gamma}(x) + N(u,v) + \gamma^{-1}\, T^{\mathfrak{D}}_{\tilde{S},\gamma}(\tilde{z}) = 0$ which is the required Cayley resolvent equation problem (8)

We can propose the following iterative approach using the fixed-point formulation (14).

Iterative Algorithm 6. Utilizing the following Scheme, determine the sequence $\{x_n\}$, $\{z_n\}$, $\{u_n\}$, and $\{v_n\}$ for every $x_0, \hat{z}_0 \in \widetilde{\mathcal{B}}, u_0 \in \tilde{P}(x_0)$, and $v_0 \in \tilde{Q}(x_0)$,

$$x_{n+1} = x_n - \delta[\hat{z}_n - \tilde{S}\left(\Re^{\mathfrak{D}}_{\tilde{S},\gamma}(\hat{z}_n)\right) + \gamma N(u_n, v_n) + \gamma C^{\mathfrak{D}}_{\tilde{S},\gamma}(x)]$$

where, $n = 0,1,2,3, \dots\dots\dots$ and $\gamma, \delta > 0$ is a constant.

Schemes 4-6 can be used to get the Cayley resolvent equation problem's existence and convergence results (8).

We propose an inertial extrapolation strategy for the Cayley resolvent equation issue (8) that will speed up the pace of convergence.

Again we rearrange the equation (10),

$$\tilde{z} = \frac{\tilde{S}(x) + \tilde{S}(x)}{2} - \gamma N(u,v) - \gamma C^{\mathfrak{D}}_{\tilde{S},\gamma}(x) \quad (15)$$

Using (15), we create the following implicit strategy for solving the Cayley resolvent equation problem.

Iterative Algorithm 7. Utilizing the following Scheme, determine the sequence $\{x_n\}$, $\{z_n\}$, $\{u_n\}$, and $\{v_n\}$ for every $x_0, \hat{z}_0 \in \widetilde{\mathcal{B}}, u_0 \in \tilde{P}(x_0)$, and $v_0 \in \tilde{Q}(x_0)$,

Jilin Daxue Xuebao (Gongxueban)/Journal of Jilin University (Engineering and Technology Edition)
ISSN: 1671-5497
E-Publication: Online Open Access
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DOI: 10.5281/zenodo.13986313

$$x_n = \Re^{\mathfrak{D}}_{S,\gamma}(\hat{z}_n)$$

and $\hat{z}_{n+1} = (1-\alpha_n)\hat{z}_n + \alpha_n[\frac{\tilde{S}(x_n)+\tilde{S}(x_{n+1})}{2} - \gamma N(u_{n+1}, v_{n+1}) - \gamma C^{\mathfrak{D}}_{\tilde{S},\gamma}(x_{n+1})]$,

where, $n = 0,1,2,3, \dots \dots \dots$ and $\gamma > 0$ is a constant $\alpha_n \in [0,1]$

Using the predictor-corrector method, we create the inertial extrapolation strategy to solve the Cayley resolvent equation issue (8)

Iterative Algorithm 8. Utilizing the following Scheme, determine the sequence $\{x_n\}$, $\{z_n\}$, $\{u_n\}$, and $\{v_n\}$ for every $x_0, \hat{z}_0 \in \widetilde{\mathcal{B}}$, $u_0 \in \tilde{P}(x_0)$, and $v_0 \in \tilde{Q}(x_0)$,

$$w_n = \hat{z}_n + e_n(\hat{z}_n - \hat{z}_{n-1}) \quad (16)$$

and $\hat{z}_{n+1} = (1-\alpha_n)\hat{z}_n + \alpha_n[\frac{\tilde{S}(\hat{z}_n)+\tilde{S}(w_n)}{2} - \gamma N(u_n, v_n) - \gamma C^{\mathfrak{D}}_{\tilde{S},\gamma}(w_n)]$ (17)

where, $\gamma > 0$ is a constant and $e_n, \alpha_n \in [0,1]$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$ and e_n is the extrapolating term. $\forall\, n \geq 1$.

5. MAIN RESULT

Initially, we discuss scheme 3, the convergence for the Cayley variational inclusion problem (1) in real Banach space. For the Cayley resolvent equation problem (8) in real *q*-uniformly smooth Banach space, we subsequently prove the convergence of scheme-8.

Theorem 1. Consider $\widetilde{\mathcal{B}}$ is real Banach spaces and $\tilde{S} : \widetilde{\mathcal{B}} \to \widetilde{\mathcal{B}}$ is a single-valued mapping such that $\tilde{S}$ is *r*-strongly accretive and $\lambda_{\tilde{S}}$ -Lipschitz continuous. Let $\mathfrak{D} : \widetilde{\mathcal{B}} \to 2^{\widetilde{\mathcal{B}}}$ be $\tilde{S}$ -accretive set-valued mapping and $\tilde{P}, \widetilde{Q} : \widetilde{\mathcal{B}} \to C(\widetilde{\mathcal{B}})$ are multi-valued mapping. Suppose that $\Re^{\mathfrak{D}}_{\tilde{S},\gamma} : \widetilde{\mathcal{B}} \to \widetilde{\mathcal{B}}$ is a generalized resolvent operator such that $\Re^{\mathfrak{D}}_{\tilde{S},\gamma}$ is $\frac{1}{r}$–Lipschitz continuous and $C^{\mathfrak{D}}_{\tilde{S},\gamma} : \widetilde{\mathcal{B}} \to \widetilde{\mathcal{B}}$ is the generalized Cayley approximation operator such that $C^{\mathfrak{D}}_{\tilde{S},\gamma}$ is λ_γ–Lipschitz continuous. Then the following condition is satisfied.

$$0 < 1 - \alpha_n + \frac{\alpha_n}{2r}\lambda_{\tilde{S}} + \frac{\alpha_n}{r}\gamma(\lambda_{N_1}\lambda_{\tilde{P}_{\mathfrak{D}}} + \lambda_{N_2}\lambda_{\tilde{Q}_{\mathfrak{D}}}) + (\frac{\alpha_n}{2r}\lambda_{\tilde{S}} + \frac{\alpha_n}{r}\gamma\lambda_c) < 1 \quad 18(A)$$

$$\lambda_{\tilde{S}} < r + \gamma\lambda_c \quad 18(B)$$

Where $\lambda_c = \frac{1}{r}(\lambda_{\tilde{S}} r + 2)$, $r \neq 0, \gamma \neq 0$.Let, $e_n, \alpha_n \in [0,1]$ e_n is the extrapolating term.$\forall\, n \geq 1$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$ as well as $\sum_{n=1}^{\infty} e_n(x_n - 2x_{n-1} + x_{n-2}) < \infty$ (19)

Then sequence $\{x_n\}$,$\{u_n\}$, and $\{v_n\}$ are generated by the iterative algorithm 3 strongly converges to the solution $x \in \widetilde{\mathcal{B}}$, $u \in \tilde{P}(x)$, and $v \in \tilde{Q}(x)$ of Cayley variational inclusion problem (1).

Jilin Daxue Xuebao (Gongxueban)/Journal of Jilin University (Engineering and Technology Edition)
ISSN: 1671-5497
E-Publication: Online Open Access
Vol: 43 Issue: 10-2024
DOI: 10.5281/zenodo.13986313

Proof: We have

$$\|x_{n+1}-x_n\| = ||\{(1-\alpha_n)x_n + \alpha_n \Re^{\mathfrak{D}}_{S,\gamma}[\tfrac{1}{2}(\tilde{S}(x_n)+\tilde{S}(w_n)) - \gamma N(u_n,v_n)$$
$$-\gamma C^{\mathfrak{D}}_{\tilde{S},\gamma}(w_n)]\} - \{(1-\alpha_n)x_{n-1} + \alpha_n \Re^{\mathfrak{D}}_{S,\gamma}[\tfrac{1}{2}(\tilde{S}(x_{n-1})$$
$$+\tilde{S}(w_{n-1})) - \gamma N(u_{n-1},v_{n-1}) - \gamma C^{\mathfrak{D}}_{\tilde{S},\gamma}(w_{n-1})]\}||$$

..

$$= ||(1-\alpha_n)(x_n - x_{n-1}) + \alpha_n\{\Re^{\mathfrak{D}}_{S,\gamma}[\tfrac{1}{2}(\tilde{S}(x_n)+\tilde{S}(w_n))$$
$$-\gamma N(u_n,v_n) - \gamma C^{\mathfrak{D}}_{\tilde{S},\gamma}(w_n)] - \Re^{\mathfrak{D}}_{\tilde{S},\gamma}[\tfrac{1}{2}(\tilde{S}(x_{n-1})+\tilde{S}(w_{n-1}))$$
$$-\gamma N(u_{n-1},v_{n-1}) - \gamma C^{\mathfrak{D}}_{\tilde{S},\gamma}(w_{n-1})]\}||$$
$$\le (1-\alpha_n)||x_n - x_{n-1}|| + \alpha_n||\Re^{\mathfrak{D}}_{S,\gamma}[\tfrac{1}{2}(\tilde{S}(x_n)+\tilde{S}(w_n))$$
$$-\gamma N(u_n,v_n) - \gamma C^{\mathfrak{D}}_{\tilde{S},\gamma}(w_n)] - \Re^{\mathfrak{D}}_{\tilde{S},\gamma}[\tfrac{1}{2}(\tilde{S}(x_{n-1})+\tilde{S}(w_{n-1}))$$
$$-\gamma N(u_{n-1},v_{n-1}) - \gamma C^{\mathfrak{D}}_{\tilde{S},\gamma}(w_{n-1})]|| \qquad (20)$$

Using Lipschitz continuity of generalized resolvent operator $R^{\mathfrak{D}}_{\tilde{S},\gamma}$, generalized Cayley approximation Operator $C^{\mathfrak{D}}_{\tilde{S},\gamma}$, from (20) we have

$$\|x_{n+1}-x_n\| \le (1-\alpha_n)||x_n - x_{n-1}|| + \tfrac{\alpha_n}{r}||\tfrac{1}{2}\big(\tilde{S}(x_n)+\tilde{S}(w_n)\big) - \tfrac{1}{2}(\tilde{S}(x_{n-1})$$
$$+\tilde{S}(w_{n-1})) - \gamma\{N(u_n,v_n) - N(u_{n-1},v_{n-1})\}$$
$$-\gamma\{C^{\mathfrak{D}}_{\tilde{S},\gamma}(w_n) - C^{\mathfrak{D}}_{\tilde{S},\gamma}(w_{n-1})\}||$$
$$\le (1-\alpha_n)||x_n - x_{n-1}|| + \tfrac{\alpha_n}{2r}||\big(\tilde{S}(x_n) - \tilde{S}(x_{n-1})\big)|| + \frac{\alpha_n}{2r}||(\tilde{S}(w_n)$$
$$-\tilde{S}(w_{n-1})|| + \frac{\alpha_n\gamma}{r}||N(u_n,v_n) - N(u_{n-1},v_{n-1})||$$
$$+\frac{\alpha_n\gamma}{r}||C^{\mathfrak{D}}_{\tilde{S},\gamma}(w_n) - C^{\mathfrak{D}}_{\tilde{S},\gamma}(w_{n-1})||$$
$$\le (1-\alpha_n)||x_n - x_{n-1}|| + \tfrac{\alpha_n}{2r}\lambda_{\tilde{S}}||x_n - x_{n-1}|| + \frac{\alpha_n}{2r}\lambda_{\tilde{S}}||w_n - w_{n-1}||$$
$$+\frac{\alpha_n}{r}\gamma\lambda_c\|w_n - w_{n-1}\| + \frac{\alpha_n\gamma}{r}||N(u_n,v_n) - N(u_{n-1},v_{n-1})|| \quad (21)$$

Using $\mathfrak{D}$ -Lipschitz continuity of N in both arguments, Then we have

$$|| N(u_n,v_n,) - N(u_{n-1},v_{n-1}) ||$$
$$= ||N(u_n,v_n,) - N(u_{n-1},v_n) + N(u_{n-1},v_n) - N(u_{n-1},v_{n-1}) ||$$
$$\le ||N(u_n,v_n) - N(u_{n-1},v_n)|| + ||N(u_{n-1},v_n) - N(u_{n-1},v_{n-1})||$$
$$\le \lambda_{N_1}||u_n - u_{n-1}|| + \lambda_{N_2}||v_n - v_{n-1}||$$
$$\le \lambda_{N_1}\mathcal{D}(\tilde{P}(x_n),\tilde{P}(x_{n-1})) + \lambda_{N_2}\mathcal{D}(\tilde{Q}(x_n),\tilde{Q}(x_{n-1}))$$

$$\leq \lambda_{N_1}\lambda_{\tilde{P}_{\mathfrak{D}}}||x_n - x_{n-1}|| + \lambda_{N_2}\lambda_{\tilde{Q}_{\mathfrak{D}}}||x_n - x_{n-1}||$$

$$\leq (\lambda_{N_1}\lambda_{\tilde{P}_{\mathfrak{D}}} + \lambda_{N_2}\lambda_{\tilde{Q}_{\mathfrak{D}}})||x_n - x_{n-1}|| \tag{22}$$

Now combining the equation (21) and (22) we get

$$\|x_{n+1} - x_n\| \leq (1-\alpha_n)||x_n - x_{n-1}|| + \frac{\alpha_n}{2r}\lambda_{\tilde{S}}||x_n - x_{n-1}|| + \frac{\alpha_n}{2r}\lambda_{\tilde{S}}||w_n - w_{n-1}||$$

$$+\frac{\alpha_n}{r}\gamma\lambda_c\|w_n - w_{n-1}\| + \frac{\alpha_n}{r}\gamma(\lambda_{N_1}\lambda_{\tilde{P}_{\mathfrak{D}}} + \lambda_{N_2}\lambda_{\tilde{Q}_{\mathfrak{D}}})||x_n - x_{n-1}||$$

$$\leq (1-\alpha_n + \frac{\alpha_n}{2r}\lambda_{\tilde{S}} + \frac{\alpha_n}{r}\gamma(\lambda_{N_1}\lambda_{\tilde{P}_{\mathfrak{D}}} + \lambda_{N_2}\lambda_{\tilde{Q}_{\mathfrak{D}}})||x_n - x_{n-1}||$$

$$+(\frac{\alpha_n}{2r}\lambda_{\tilde{S}} + \frac{\alpha_n}{r}\gamma\lambda_c)\|w_n - w_{n-1}\| \tag{23}$$

Using (6), we get

$$\|w_n - w_{n-1}\| = \|\{x_n + e_n(x_n - x_{n-1})\} - \{x_{n-1} + e_n(x_{n-1} - x_{n-2})\}\|$$

$$= \|x_n + e_n(x_n - x_{n-1}) - x_{n-1} - e_n(x_{n-1} - x_{n-2})\|$$

$$= \|x_n - x_{n-1} + e_n(x_n - 2x_{n-1} + x_{n-2})\|$$

$$\leq \|x_n - x_{n-1}|| + e_n||x_n - 2x_{n-1} + x_{n-2}\| \tag{24}$$

we have from (23) and (24)

$$\|x_{n+1} - x_n\| \leq (1-\alpha_n + \frac{\alpha_n}{2r}\lambda_{\tilde{S}} + \frac{\alpha_n}{r}\gamma(\lambda_{N_1}\lambda_{\tilde{P}_{\mathfrak{D}}} + \lambda_{N_2}\lambda_{\tilde{Q}_{\mathfrak{D}}})||x_n - x_{n-1}||$$

$$+(\frac{\alpha_n}{2r}\lambda_{\tilde{S}} + \frac{\alpha_n}{r}\gamma\lambda_c)\{\|x_n - x_{n-1}\| + e_n||x_n - 2x_{n-1} + x_{n-2}||\}$$

$$\leq \{1-\alpha_n + \frac{\alpha_n}{2r}\lambda_{\tilde{S}} + \frac{\alpha_n}{r}\gamma(\lambda_{N_1}\lambda_{\tilde{P}_{\mathfrak{D}}} + \lambda_{N_2}\lambda_{\tilde{Q}_{\mathfrak{D}}}) + (\frac{\alpha_n}{2r}\lambda_{\tilde{S}} + \frac{\alpha_n}{r}\gamma\lambda_c)\}$$

$$\|x_n - x_{n-1}\| + (\frac{\alpha_n}{2r}\lambda_{\tilde{S}} + \frac{\alpha_n}{r}\gamma\lambda_c)e_n||x_n - 2x_{n-1} + x_{n-2}||$$

$$\leq \theta_1(x)\|x_n - x_{n-1}\| + \theta_2(x)e_n||x_n - 2x_{n-1} + x_{n-2}|| \tag{25}$$

Where, $\theta_1(x) = 1-\alpha_n + \frac{\alpha_n}{2r}\lambda_{\tilde{S}} + \frac{\alpha_n}{r}\gamma(\lambda_{N_1}\lambda_{\tilde{P}_{\mathfrak{D}}} + \lambda_{N_2}\lambda_{\tilde{Q}_{\mathfrak{D}}}) + (\frac{\alpha_n}{2r}\lambda_{\tilde{S}} + \frac{\alpha_n}{r}\gamma\lambda_c)$ and $\theta_2(x) = \frac{\alpha_n}{2r}\lambda_{\tilde{S}} + \frac{\alpha_n}{r}\gamma\lambda_c$

Let us consider $0 < \theta_1(x) < 1$ and $0 < \theta_2(x) < 1$, From condition 18(A) and 18(B),

By condition (19), We have, $\sum_{n=1}^{\infty} e_n||x_n - 2x_{n-1} + x_{n-2}|| < \infty$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$, Again, consider $\sigma_n = 0$ and $\tilde{\gamma}_n = \sum_{n=1}^{\infty} e_n||x_n - 2x_{n-1} + x_{n-2}|| < \infty$. Utilize lemma 2, and (25) we get, $x_n \to x$ as $n \to \infty$ and also, we get $u_n \to u$ as $n \to \infty$, and $v_n \to v$ as $n \to \infty$

Theorem 2. Consider $\widetilde{\mathcal{B}}$ is q-uniformly smooth Banach space and $\tilde{S} : \widetilde{\mathcal{B}} \to \widetilde{\mathcal{B}}$ is a single-valued mapping such that $\tilde{S}$ is one-one, $\lambda_{\tilde{S}}$ Lipschitz continuous, $\beta_{\tilde{S}}$ expansive and r-strongly accretive. and $\mathfrak{D} : \widetilde{\mathcal{B}} \to 2^{\widetilde{\mathcal{B}}}$ is $\tilde{S}$ accretive set-valued mapping and $\tilde{P}, \tilde{Q} : \widetilde{\mathcal{B}} \to C(\widetilde{\mathcal{B}})$ are multi-valued mapping. Then the generalized resolvent operator $\mathfrak{R}_{\tilde{S},\gamma}^{\mathfrak{D}}$:

Jilin Daxue Xuebao (Gongxueban)/Journal of Jilin University (Engineering and Technology Edition)
ISSN: 1671-5497
E-Publication: Online Open Access
Vol: 43 Issue: 10-2024
DOI: 10.5281/zenodo.13986313

$\widetilde{\mathcal{B}} \to \widetilde{\mathcal{B}}$ is Lipschitz continuous with constant $\frac{1}{r}$. Let $C^{\mathfrak{D}}_{\tilde{S},\gamma} : \widetilde{\mathcal{B}} \to \widetilde{\mathcal{B}}$ be a generalized Cayley approximation operator Such that $C^{\mathfrak{D}}_{\tilde{S},\gamma}$ is θ_c- strongly accretive concerning $\tilde{S}$ and λ_c - Lipschitz continuous. Let $T^{\mathfrak{D}}_{\tilde{S},\gamma}(\tilde{z}) = [I - \tilde{S}(\mathfrak{R}^{\mathfrak{D}}_{\tilde{S},\gamma})](\tilde{z})$, Where $\tilde{S}\left[\mathfrak{R}^{\mathfrak{D}}_{\tilde{S},\gamma}(\tilde{z})\right] = \left[\tilde{S}\mathfrak{R}^{\mathfrak{D}}_{\tilde{S},\gamma}\right](\tilde{z}),\ \tilde{z} \in \widetilde{\mathcal{B}}$.

Let us consider the following postulate is satisfied

$$0 < 1 - \alpha_n + \tfrac{\alpha_n}{2}\lambda_{\tilde{S}} - \alpha_n\gamma\left(\lambda_{N_1}\lambda_{\tilde{P}_{\mathfrak{D}}} + \lambda_{N_2}\lambda_{\tilde{Q}_{\mathfrak{D}}}\right) + \tfrac{\alpha_n}{2}\sqrt[q]{\left(\lambda_{\tilde{S}}{}^q - 2q\gamma\theta_c + 2^q C_q\gamma^q\lambda_c{}^q\right)} < 1 \quad (26)$$

$\lambda_{\tilde{S}} < 2$
(27)

where,$\gamma > 0$ is a constant and $e_n, \alpha_n \in [0,1]$ such that $\sum_{n=1}^{\infty} e_n||\hat{z}_n - 2\hat{z}_{n-1} + \hat{z}_{n-2}|| < \infty$
(28)

and $\sum_{n=1}^{\infty}\alpha_n = \infty$,$\theta_c = \frac{1}{r\gamma}(\beta_{\tilde{S}}{}^q - \lambda_{\tilde{S}}{}^{q-1})$, $\lambda_c = \frac{1}{r}(\lambda_{\tilde{S}}r + 2)$, $r\gamma \neq 0$,$r \neq 0$, $\beta_{\tilde{S}}{}^q > \lambda_{\tilde{S}}{}^{q-1}$

If all the constants are positive,Then sequence $\{x_n\}$, $\{z_n\}$ $\{u_n\}$ and $\{v_n\}$ generated by iterative algorithm 8 strongly convergence to the unique solution x, z, u and v of Cayley resolvent equation problem (8).

Proof: Applying (17) of scheme-8 and Lipschitz continuity of $\tilde{S}$, We evaluate

$$\begin{aligned}
\|\hat{z}_{n+1} - \hat{z}_n\| &= ||\,[(1-\alpha_n)\hat{z}_n + \alpha_n\{\tfrac{1}{2}(\tilde{S}(\hat{z}_n) + \tilde{S}(w_n)) - \gamma N(u_n, v_n) \\
&\quad -\gamma C^{\mathfrak{D}}_{\tilde{S},\gamma}(w_n)\}] - [(1-\alpha_n)\hat{z}_{n-1} + \alpha_n\{\tfrac{1}{2}(\tilde{S}(\hat{z}_{n-1}) + \tilde{S}(w_{n-1})) \\
&\quad -\gamma N(u_{n-1}, v_{n-1}) - \gamma C^{\mathfrak{D}}_{\tilde{S},\gamma}(w_{n-1})\}]|| \\
&\le (1-\alpha_n)||\hat{z}_n - \hat{z}_{n-1}|| + \tfrac{\alpha_n}{2}||(\tilde{S}(\hat{z}_n) - \tilde{S}(\hat{z}_{n-1})|| - \alpha_n\gamma \\
&\quad ||N(u_n, v_n) - N(u_{n-1}, v_{n-1})|| + \tfrac{\alpha_n}{2}||(\tilde{S}(w_n) - \tilde{S}(w_{n-1}) \\
&\quad -2\gamma\{C^{\mathfrak{D}}_{\tilde{S},\gamma}(w_n) - C^{\mathfrak{D}}_{\tilde{S},\gamma}(w_{n-1})\}|| \\
&\le (1-\alpha_n)||\hat{z}_n - \hat{z}_{n-1}|| + \tfrac{\alpha_n}{2}\lambda_{\tilde{S}}||\hat{z}_n - \hat{z}_{n-1}|| - \alpha_n\gamma(\lambda_{N_1}\lambda_{\tilde{P}_{\mathfrak{D}}} \\
&\quad +\lambda_{N_2}\lambda_{\tilde{Q}_{\mathfrak{D}}})\,||\hat{z}_n - \hat{z}_{n-1}|| + \tfrac{\alpha_n}{2}||(\tilde{S}(w_n) - \tilde{S}(w_{n-1}) - 2\gamma \\
&\quad \{C^{\mathfrak{D}}_{\tilde{S},\gamma}(w_n) - C^{\mathfrak{D}}_{\tilde{S},\gamma}(w_{n-1})\}|| \qquad \text{[By (25)]} \\
&\le \{1 - \alpha_n + \tfrac{\alpha_n}{2}\lambda_{\tilde{S}} - \alpha_n\gamma(\lambda_{N_1}\lambda_{\tilde{P}_{\mathfrak{D}}} + \lambda_{N_2}\lambda_{\tilde{Q}_{\mathfrak{D}}})\}||\hat{z}_n - \hat{z}_{n-1}|| \\
&\quad + \tfrac{\alpha_n}{2}||(\tilde{S}(w_n) - \tilde{S}(w_{n-1}) - 2\gamma\{C^{\mathfrak{D}}_{\tilde{S},\gamma}(w_n) - C^{\mathfrak{D}}_{\tilde{S},\gamma}(w_{n-1})\}|| \qquad (29)
\end{aligned}$$

Using Lemma -1 and Lipschitz continuity of $\tilde{S}$, Strongly convergence and Lipschitz continuity of $C^{\mathfrak{D}}_{\tilde{S},\gamma}$ associate $\tilde{S}$.we have,

Jilin Daxue Xuebao (Gongxueban)/Journal of Jilin University (Engineering and Technology Edition)
ISSN: 1671-5497
E-Publication: Online Open Access
Vol: 43 Issue: 10-2024
DOI: 10.5281/zenodo.13986313

$$\left\|\tilde{S}(w_n) - \tilde{S}(w_{n-1}) - 2\gamma[C^{\mathfrak{D}}_{\tilde{S},\gamma}(w_n) - C^{\mathfrak{D}}_{\tilde{S},\gamma}(w_{n-1})]\right\|^q$$
$$\leq \left\|\tilde{S}(w_n) - \tilde{S}(w_{n-1})\right\|^q - 2q\gamma\langle C^{\mathfrak{D}}_{\tilde{S},\gamma}(w_n) - C^{\mathfrak{D}}_{\tilde{S},\gamma}(w_{n-1}), \mathcal{N}_q\{\tilde{S}(w_n) - \tilde{S}(w_{n-1})\}\rangle$$
$$+ 2^q C_q \gamma^q \left\|C^{\mathfrak{D}}_{\tilde{S},\gamma}(w_n) - C^{\mathfrak{D}}_{\tilde{S},\gamma}(w_{n-1})\right\|^q$$
$$\leq \lambda_{\tilde{S}}{}^q \|w_n - w_{n-1}\|^q - 2q\gamma\theta_c \|w_n - w_{n-1}\|^q + 2^q C_q \gamma^q$$
$$\left\|C^{\mathfrak{D}}_{\tilde{S},\gamma}(w_n) - C^{\mathfrak{D}}_{\tilde{S},\gamma}(w_{n-1})\right\|^q$$
$$\leq (\lambda_{\tilde{S}}{}^q - 2q\gamma\theta_c)\|w_n - w_{n-1}\|^q + 2^q C_q \gamma^q \lambda_c{}^q \|w_n - w_{n-1}\|^q$$
$$\leq (\lambda_{\tilde{S}}{}^q - 2q\gamma\theta_c + 2^q C_q \gamma^q \lambda_c{}^q)\|w_n - w_{n-1}\|^q \quad (30)$$

If follows from (30) then we have

$$||\tilde{S}(w_n) - \tilde{S}(w_{n-1}) - 2\gamma[C^{\mathfrak{D}}_{\tilde{S},\gamma}(w_n) - C^{\mathfrak{D}}_{\tilde{S},\gamma}(w_{n-1})]||$$
$$\leq \sqrt[q]{\left(\lambda_{\tilde{S}}{}^q - 2q\gamma\theta_c + 2^q C_q \gamma^q \lambda_c{}^q\right)}||w_n - w_{n-1}|| \quad (31)$$

Combining (29) and (31), We get,

$$\|\hat{z}_{n+1} - \hat{z}_n\| \leq \{1 - \alpha_n + \tfrac{\alpha_n}{2}\lambda_{\tilde{S}} - \alpha_n\gamma(\lambda_{N_1}\lambda_{\tilde{P}_{\mathfrak{D}}} + \lambda_{N_2}\lambda_{\tilde{Q}_{\mathfrak{D}}})\}||\hat{z}_n - \hat{z}_{n-1}||$$
$$+ \frac{\alpha_n}{2}\sqrt[q]{\left(\lambda_{\tilde{S}}{}^q - 2q\gamma\theta_c + 2^q C_q \gamma^q \lambda_c{}^q\right)}||w_n - w_{n-1}|| \quad (32)$$

Applying (16) of scheme 8, we have

$$\|w_n - w_{n-1}\| = \|\{\hat{z}_n + e_n(\hat{z}_n - \hat{z}_{n-1})\} - \{\hat{z}_{n-1} + e_n(\hat{z}_{n-1} - \hat{z}_{n-2})\}\|$$
$$\leq \|\hat{z}_n - \hat{z}_{n-1}|| + e_n||\hat{z}_n - 2\hat{z}_{n-1} + \hat{z}_{n-2}\| \quad (33)$$

Combining (32) with (33), We have

$$\|\hat{z}_{n+1} - \hat{z}_n\| \leq \{1 - \alpha_n + \tfrac{\alpha_n}{2}\lambda_{\tilde{S}} - \alpha_n\gamma(\lambda_{N_1}\lambda_{\tilde{P}_{\mathfrak{D}}} + \lambda_{N_2}\lambda_{\tilde{Q}_{\mathfrak{D}}})\}||\hat{z}_n - \hat{z}_{n-1}||$$
$$+ \frac{\alpha_n}{2}\sqrt[q]{\left(\lambda_{\tilde{S}}{}^q - 2q\gamma\theta_c + 2^q C_q \gamma^q \lambda_c{}^q\right)}\{||\hat{z}_n - \hat{z}_{n-1}||$$
$$+e_n||\hat{z}_n - 2\hat{z}_{n-1} + \hat{z}_{n-2}||$$

$$\leq \{1 - \alpha_n + \tfrac{\alpha_n}{2}\lambda_{\tilde{S}} - \alpha_n\gamma\left(\lambda_{N_1}\lambda_{\tilde{P}_{\mathfrak{D}}} + \lambda_{N_2}\lambda_{\tilde{Q}_{\mathfrak{D}}}\right)$$
$$+ \frac{\alpha_n}{2}\sqrt[q]{\left(\lambda_{\tilde{S}}{}^q - 2q\gamma\theta_c + 2^q C_q \gamma^q \lambda_c{}^q\right)}\}||\hat{z}_n - \hat{z}_{n-1}||$$
$$+ \frac{\alpha_n}{2}\sqrt[q]{\left(\lambda_{\tilde{S}}{}^q - 2q\gamma\theta_c + 2^q C_q \gamma^q \lambda_c{}^q\right)}\, e_n||\hat{z}_n - 2\hat{z}_{n-1} + \hat{z}_{n-2}||$$

ISSN: 1671-5497
E-Publication: Online Open Access
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DOI: 10.5281/zenodo.13986313

Thus we have,

$$\|\hat{z}_{n+1} - \hat{z}_n\| \leq \theta_3(z)||\,\hat{z}_n - \hat{z}_{n-1}|| + \theta_4(z)e_n||\hat{z}_n - 2\hat{z}_{n-1} + \hat{z}_{n-2}|| \quad (34)$$

Wher$\theta_3(z) = 1 - \alpha_n + \frac{\alpha_n}{2}\lambda_{\tilde{S}} - \alpha_n\gamma\left(\lambda_{N_1}\lambda_{\tilde{P}_{\mathfrak{D}}} + \lambda_{N_2}\lambda_{\tilde{Q}_{\mathfrak{D}}}\right) + \frac{\alpha_n}{2}\sqrt[q]{\left(\lambda_{\tilde{S}}{}^q - 2q\gamma\theta_c + 2^q C_q\gamma^q\lambda_c{}^q\right)}$,

$\theta_4(z) = \frac{\alpha_n}{2}\sqrt[q]{\left(\lambda_{\tilde{S}}{}^q - 2q\gamma\theta_c + 2^q C_q\gamma^q\lambda_c{}^q\right)}$ and also $\lambda_{\tilde{S}}{}^q + 2^q C_q\gamma^q\lambda_c{}^q > 2q\gamma\theta_c$

$\theta_c = \frac{1}{r\gamma}(\beta_{\tilde{S}}{}^q - \lambda_{\tilde{S}}{}^{q-1})$, $\lambda_c = \frac{1}{r}(\lambda_{\tilde{S}}r + 2)$, $r\gamma \neq 0$,$r \neq 0$, $\beta_{\tilde{S}}{}^q > \lambda_{\tilde{S}}{}^{q-1}$

Again, utilizing condition (28),we get $\sum_{n=1}^{\infty}\alpha_n = \infty$ and $\sum_{n=1}^{\infty} e_n||\hat{z}_n - 2\hat{z}_{n-1} + \hat{z}_{n-2}|| < \infty$, Letting $\sigma_n = 0$ and $\hat{\gamma}_n = \sum_{n=1}^{\infty} e_n||\hat{z}_n - 2\hat{z}_{n-1} + \hat{z}_{n-2}|| < \infty$,Then from (34) and lemma 2 we have $\hat{z}_n \to z \in \widetilde{\mathcal{B}}$, as $n \to \infty$.

Since $\hat{z}_n \to z \in \widetilde{\mathcal{B}}$, then (34) implies that $x_n \to x \in \widetilde{\mathcal{B}}$, $u_n \to u \in \tilde{P}(x)$ and $v_n \to v \in \tilde{Q}(x)$.

Therefore, the sequence $\{x_n\}$, $\{\hat{z}_n\}$, $\{u_n\}$, and $\{v_n\}$ represented by scheme 8 strongly convergence to the solutions x ,z , u and v of the Cayley resolvent equation problem (8).That is, $x_n \to x$ as $n \to \infty$ and also we get $u_n \to u$ as $n \to \infty$, $v_n \to v$ as $n \to \infty$.

6. MATHEMATICAL EXPERIMENT

Let $\widetilde{\mathcal{B}} = \mathbb{R}$ with the usual inner product and norm, $\tilde{S} : \widetilde{\mathcal{B}} \to \widetilde{\mathcal{B}}$ be single-valued mapping and $\mathfrak{D} : \widetilde{\mathcal{B}} \to 2^{\widetilde{\widetilde{\mathcal{B}}}}$ be set-valued mapping such that

$$\tilde{S}(x) = \frac{8}{7}x$$

And $\mathfrak{D}(x) = \left\{\frac{1}{14}x\right\}$, for all $x \in \widetilde{\mathcal{B}}$

Suppose, $N : \widetilde{\mathcal{B}} \times \widetilde{\mathcal{B}} \to \widetilde{\mathcal{B}}$ be the single-valued mappings and $\tilde{S}, \tilde{T} : \widetilde{\mathcal{B}} \to C(\widetilde{\mathcal{B}})$ are multi-valued mapping such that

$$P(x) = \{\tfrac{x}{7}\}$$

and $Q(x) = \{\frac{x}{6}\}$

$$N(u, v) = \{\,\tfrac{u}{2} + \tfrac{v}{2}\,\}$$

Now we have

$$\begin{aligned} D\big(P(x), P(y)\big) &= \max\{sup_{x\in P(x)}d\big(x, F(y)\big), sup_{y\in P(y)}d(F(x), y)\} \\ &\leq \max\{||\tfrac{x}{7} - \tfrac{y}{7}||, ||\tfrac{y}{7} - \tfrac{x}{7}||\} \\ &\leq \tfrac{1}{7}\max\{||x - y||, ||y - x||\} \\ &\leq \tfrac{1}{5}||x - y|| \end{aligned}$$

Thus *P* is *D*-Lipschitz continuous with constant $\lambda_{\tilde{P}_{\mathfrak{D}}} = \frac{1}{5}$, Similarly, we have to show that $\lambda_{\tilde{Q}_{\mathfrak{D}}} = \frac{1}{3}$

Jilin Daxue Xuebao (Gongxueban)/Journal of Jilin University (Engineering and Technology Edition)
ISSN: 1671-5497
E-Publication: Online Open Access
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DOI: 10.5281/zenodo.13986313

Hence N is Lipschitz continuous in both arguments with constants $\lambda_{N_1} = \lambda_{N_2} = 1$ and $N(u,v) = \frac{x}{14} + \frac{x}{12} = \frac{13}{84}x$

(i) $\tilde{S}$ is *r*- strongly accretive and $\lambda_{\tilde{S}}$ – Lipschitz continuous.

$$\langle \tilde{S}(x) - \tilde{S}(y), x - y \rangle = \langle \frac{8}{7}x - \frac{8}{7}y, x - y \rangle$$
$$= \frac{8}{7}\| x - y\|^2$$
$$\geq \frac{15}{14}\|x - y\|^2$$

Thus, $\tilde{S}$ is $r = \frac{15}{14}$- strongly accretive mapping.

and $\|\tilde{S}(x) - \tilde{S}(y)\| = \| \frac{8}{7}x - \frac{8}{7}y \|$
$$= \frac{8}{7}\|x - y\|$$
$$\leq \frac{17}{14}\|x - y\|$$

Thus $\tilde{S}$ is $\lambda_{\tilde{S}} = \frac{17}{14}$ -Lipschitz continuous.

(ii) $\mathfrak{D}$ is $\tilde{S}$ is accretive

$$\|\mathfrak{D}(x) - \mathfrak{D}(y)\| = \left\|\frac{1}{14}x - \frac{1}{14}y\right\| = \frac{1}{14}\|x - y\| \geq 0$$

That is, $\mathfrak{D}$ is accretive and also for $\gamma = 1$, it is easy to verify that

$$[\tilde{S} + \gamma\mathfrak{D}](\widetilde{\mathcal{B}}) = \widetilde{\mathcal{B}}$$

For $\gamma = 1$, we define a generalized resolvent operator as

$$\mathfrak{R}^{\mathfrak{D}}_{\tilde{S},\gamma}(x) = [\tilde{S} + \gamma\mathfrak{D}]^{-1}(x) = \frac{14}{17}x$$

And $|| \mathfrak{R}^{\mathfrak{D}}_{\tilde{S},\gamma}(x) - \mathfrak{R}^{\mathfrak{D}}_{\tilde{S},\gamma}(y)|| = || \frac{14}{17}x - \frac{14}{17}y ||$
$$= \frac{14}{17}\|x - y\|$$
$$\leq \frac{14}{15}\|x - y\|$$
$$\leq \frac{1}{(^{15}/_{14})}\|x - y\|$$

Thus the generalized resolvent operator $\mathfrak{R}^{\mathfrak{D}}_{\tilde{S},\gamma}$ is $\frac{1}{r} = \frac{1}{(^{15}/_{14})}$ Lipschitz continuous.

(iii) Calculate the generalized Cayley approximation operator

$$C^{\mathfrak{D}}_{\tilde{S},\gamma}(x) = [2\,\mathfrak{R}^{\mathfrak{D}}_{\tilde{S},\gamma} - \tilde{S}](x) = \frac{60}{119}x,\ x \in \widetilde{\mathcal{B}}$$

and $\|C^{\mathfrak{D}}_{\tilde{S},\gamma}(x) - C^{\mathfrak{D}}_{\tilde{S},\gamma}(y)\| = \| \frac{60}{119}x - \frac{60}{119}y \|$
$$= \frac{60}{119}\| x - y \|$$

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ISSN: 1671-5497
E-Publication: Online Open Access
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DOI: 10.5281/zenodo.13986313

$$\leq \frac{647}{210} \| x - y \|$$

Thus the generalized Cayley approximation operator $C^{\mathfrak{D}}_{\tilde{S},\gamma}$ is $\lambda_c = \frac{r\lambda_{\tilde{S}}+2}{r} = \frac{647}{210}$ − Lipschitz continuous.

(iv) Considering the constants calculated above, the conditions 18(A),18(B),(26) and (27) of theorem -1 and theorem-2 are satisfied.

(v) Using Iterative Algorithm-3 and an inertial extrapolation scheme where, $e_n = \frac{1}{n+1}$ and $\alpha_n = \frac{1}{n}$,

We get, $w_n = x_n + e_n(x_n - x_{n-1})$

$$\text{and } x_{n+1} = (1-\alpha_n)x_n + \alpha_n \Re^{\mathfrak{D}}_{\tilde{S},\gamma}\left[\frac{\tilde{S}(x_n) + \tilde{S}(w_n)}{2} - \gamma N(u_n, v_n) - \gamma C^{\mathfrak{D}}_{\tilde{S},\gamma}(w_n)\right]$$

Then we have, $x_{n+1} = (1-\alpha_n)x_n + \alpha_n \left[\frac{35}{102} x_n + \frac{112}{2023} w_n\right]$

Let us consider the various initial values. $x_0 = 4.0,\ 2.0,\ 1.0, -1.0, -2.0, -4.0$, Now, using MATLAB R2024a. we get the estimation table (taken up to four decimal places) and convergence graph and observe that the sequence x_n converges at $x = 0$, which is the solution of the Cayley variational inclusion problem (1).

Estimation Table

No. of Iterations	$x_0 = 4.0$ x_n	$x_0 = 2.0$ x_n	$x_0 = 1.0$ x_n	$x_0 = -1.0$ x_n	$x_0 = -2.0$ x_n	$x_0 = -4.0$ x_n
1	1.7900	0.8949	0.4474	-0.4474	-0.8949	- 1.7900
2	0.8009	0.4005	0.2002	-0.2002	-0.4005	-0.8009
3	0.3584	0.1792	0.0896	-0.0896	-0.1792	-0.3584
4	0.1604	0.0801	0.0400	-0.0400	-0.0801	-0.1604
5	0.0717	0.0358	0.0179	-0.0179	-0.0358	-0.0717
6	0.0321	0.0160	0.0080	-0.0080	-0.0160	-0.0321
7	0.0143	0.0071	0.0035	-0.0035	-0.0071	-0.0143
8	0.0064	0.0032	0.0016	-0.0016	-0.0032	-0.0064
9	0.0028	0.0014	0.0007	-0.0007	-0.0014	-0.0028
10	0.0012	0.0006	0.0003	-0.0003	-0.0006	-0.0012
11	0.0005	0.0002	0.0001	-0.0001	-0.0002	-0.0005
12	0.0002	0.0001	6.44e-05	-6.44e-05	-0.0001	-0.0002
15	2.311e-05	1.155e-05	5.777e-06	-5.777e-06	-1.15e-05	-2.31e-05
20	4.147e-07	2.073e-07	1.036e-07	-1.036e-07	-2.07e-07	-4.147e-07
25	7.4416e-09	3.720e-09	1.860e-09	-1.860e-09	-3.720e-09	-7.441e-09
30	2.984e-10	1.492e-10	7.460e-11	-7.460e-11	-1.492e-10	-2.984e-10

Jilin Daxue Xuebao (Gongxueban)/Journal of Jilin University (Engineering and Technology Edition)
ISSN: 1671-5497
E-Publication: Online Open Access
Vol: 43 Issue: 10-2024
DOI: 10.5281/zenodo.13986313

Graphical Interpretation

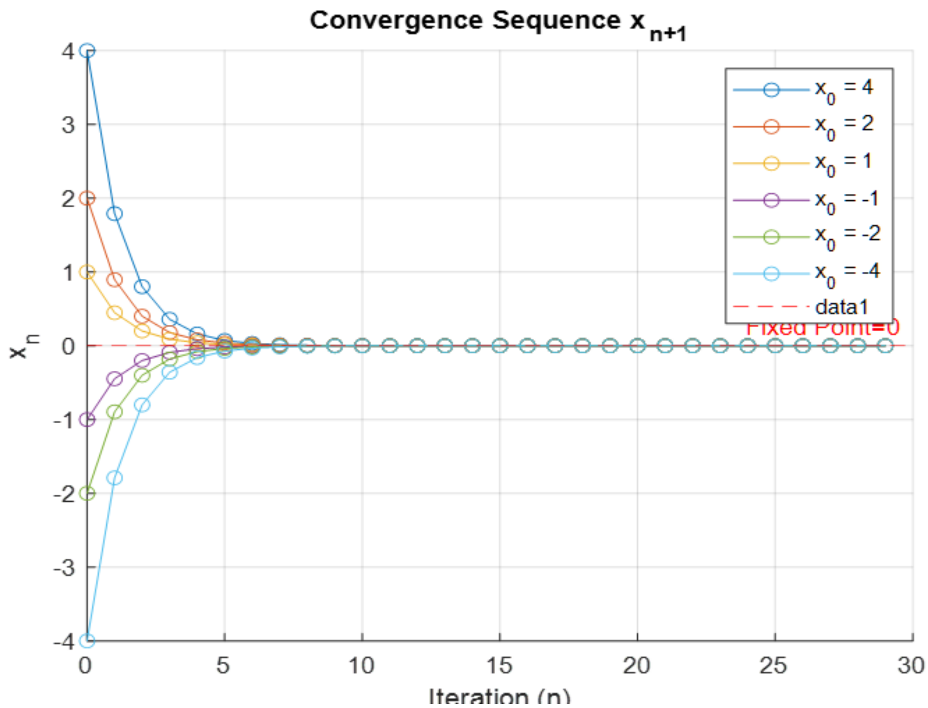

Figure: convergence sequence x_{n+1}

CONCLUSION

This work examines the Cayley variational inclusion issue as well as its corresponding Cayley resolvent equation problem. Both issues can be solved by employing the resolvent operator technique, subject to specific criteria. Our paper primarily examines the convergence analysis for both problems utilizing the inertial extrapolation scheme. Implementing an inertial extrapolation method in both techniques of a mathematical model proves that the convergence is relatively rapid, as indicated by the convergence graph quickly approaching zero. One can enhance our results in spaces with a more significant number of dimensions. Our results can be applied by engineers, physicists, and other researchers for practical purposes.

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Jilin Daxue Xuebao (Gongxueban)/Journal of Jilin University (Engineering and Technology Edition)
ISSN: 1671-5497
E-Publication: Online Open Access
Vol: 43 Issue: 10-2024
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