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TEMPERATURE BASED INVARIANTS FOR BRIDGE GRAPHS

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Abstract

This paper investigates the application of topological indices to bridge graphs, which are graphs where every edge is a bridge. Topological indices are numerical measures that provide insights into the structural properties of graphs. Understanding the behavior of these indices in the context of bridge graphs is important for unraveling their unique characteristics and connectivity patterns. We will apply the concept of temperature based topological indices on different unknown topological indices namely, 1^{st} and 2^{nd} Zagreb indices, Forgotten index, productivity index and reciprocal productivity index. This study demonstrates that temperature based topological indices exhibit distinct patterns when applied to bridge graph over path, cycle and complete graph. Some indices effectively capture the bridge structure and connectivity properties, while others may not provide meaningful information. The limitations and challenges associated with using topological indices for bridge graphs are discussed, emphasizing the need for new or modified indices that better capture the unique characteristics of this graph type.

Keywords: Topological Indices, Temperature Based Invariants, Bridge Graph over Path, Bridge Graph over Cycle, Bridge Graph over Complete Graph.

1. INTRODUCTION

Graph theory plays a fundamental role in various disciplines, ranging from computer science to chemistry and biology. Topological indices are numerical parameters that capture structural information about a graph, providing insights into its characteristics and properties. In this paper, we present a comprehensive study on the computation of topological indices for bridge graphs, specifically focusing on bridge graphs constructed from path, cycle, and complete graph structures [1]. The bridge graph is an important class of graphs that arises when edges are added to an underlying graph, connecting two nonadjacent vertices. By introducing these additional edges, the bridge graph exhibits unique properties and structural characteristics, making it an intriguing subject of investigation. Through our research, we aim to explore the topological indices of bridge graphs and provide a deeper understanding of their mathematical properties [2]. To accomplish our objective, we first define the bridge graph over a path, cycle, and complete graph. We then proceed to compute and analyze various topological indices associated with these bridge graphs. Specifically, we focus on indices such as the Wiener index, Zagreb indices, Randić index, and other commonly used measures. By employing established formulas and algorithms, we calculate these indices and investigate their behavior as the bridge graph structures evolve [3]. Our computational analysis reveals intriguing patterns and trends in the computed topological indices. We observe that the addition of bridges significantly influences the values of these indices, indicating the impact of structural modifications on the overall graph properties. Through extensive experimentation and analysis, we aim to provide a comprehensive understanding of the relationships between bridge graphs and topological indices, facilitating future research and applications in various domains [4]. Overall, this paper contributes to the field of graph theory by investigating the computation of topological indices for bridge graphs over path, cycle, and complete graph structures. The insights gained from our study enhance our understanding of the interplay between graph structures and topological indices, offering valuable knowledge for graph theoreticians, computational chemists, and researchers working in related disciplines.

2. LITERATURE REVIEW

A graph $G = (V, E)$ is an order pair, where V represents the vertex set and E represents the edge set. Chemical graph theory is about the discussion of chemical structure by means of graphs. Conversion of chemical structure can be made by applying the definition of chemical graph theory, where atoms and bonds refers to vertices and edges, respectively. [5, 6, 7] We can transform chemical structure into graph, compute desire result and then go back to original problem, with solution. This is the beauty of chemical graph theory. A graph having no loop or multiple edge in known as simple graph [8]. A molecular graph is a simple graph in which atoms and bounds are represented by vertex and edge set respectively. The degree of vertex is the number of edges attached with that vertex [9]. These properties of various objects is of primary interest. Winner, in 1947, introduce the concept of first topological index while finding the boiling point. In 1975, Gutman gave a remarkable identity [10] about Zagreb indices. Hence, these two indices are among the oldest degree-based descriptors and their properties are extensively investigated. The mathematical formulae of these indices are:

$$M_1(G) = \sum_{uv \epsilon E(G)} (du + dv),$$

$$M_2(G) = \sum_{uv \epsilon E(G)} (du \times dv)$$

Furtula and Gutman, introduce the concept of forgotten index [11]

$$F(G) = \sum_{uv \epsilon E(G)} (du^2 + dv^2)$$

Product connectivity index and its reciprocal version is defined as [12]

$$P(G) = \sum_{uv \epsilon E(G)} \frac{1}{\sqrt{d_u d_v}}$$

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$$RP(G) = \sum_{uv\epsilon E(G)} \sqrt{d_u d_v}$$

Kulli, introduce the idea of temperature based topological indices [13]

$$T_u(G) = \frac{T_u}{n - T_u} \qquad n = |V(G)|$$

Motivated by the work of kulli, we apply the concept of Temperature based version of above mentioned topological indices are

$$TM_1(G) = \sum_{uv\epsilon E(G)} (Tu + Tv),$$

$$TM_2(G) = \sum_{uv\epsilon E(G)} (Tu \times Tv)$$

$$TF(G) = \sum_{uv\epsilon E(G)} (Tu^2 + Tv^2)$$

$$TP(G) = \sum_{uv\epsilon E(G)} \frac{1}{\sqrt{T_u T_v}}$$

$$TRP(G) = \sum_{uv\epsilon E(G)} \sqrt{T_u T_v}$$

2.1 *Bridge graph over path* $G_l(P_m, v)$ *over* p_m, $m > 2$

We can observe the vertex set V, from Figure 1, we can divide the vertex set into four subsets V_1, V_2, V_3, and V_4, Such that $V = V_1 + V_2 + V_3 + V_4$.

If E represents the edge set. The Figure 1 shows that there are four different types of edges present in the graph of Bridge graph $G_l\ (P_m, v)$ over Path P_m.

The Table 1, explain in detail the partition of edge set and their temperature based invariants [11].

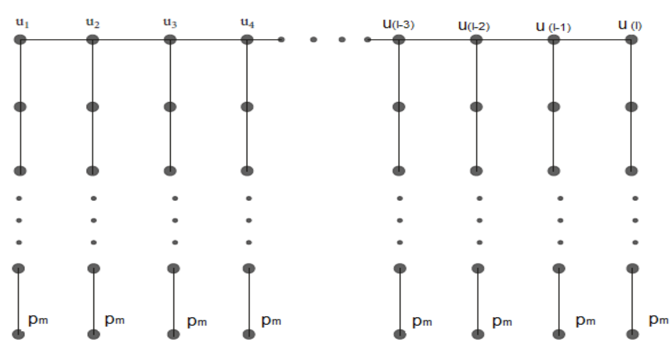

Figure 1: Bridge graph over p_m

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$(d_u, d_v$	$\frac{d_u}{n-d_u}$	$\frac{d_v}{n-d_v}$	$Recurrence$
(1,2)	$\frac{1}{lm-1}$	$\frac{2}{lm-2}$	l
(2,2)	$\frac{2}{lm-2}$	$\frac{2}{lm-2}$	$3l+2$
(2,3)	$\frac{2}{lm-2}$	$\frac{3}{lm-3}$	l
(3,3)	$\frac{3}{lm-3}$	$\frac{3}{lm-3}$	$l-3$

Table1: Edge partition of Bridge graph over p_m

Theorem 1

$Let\ G\ be\ a\ graph\ of\ \boldsymbol{G_l(P_m, v)\ over\ p_m\ then\ its\ temperature - based\ indices\ are}$

- $TM_1(G) = \frac{25l^3m^2-10l^2m^2-92l^2m+22lm+69l-12}{(lm-1)(lm-2)(lm-3)}$
- $TM_2(G) = \frac{28l^4m^3-19l^3m^3-173l^3m^2+79l^2m^2+339l^2m-96lm-198l+36}{(lm-1)(lm-2)^2(lm-3)^2}$
- $TF(G) = \frac{57l^5m^4-38l^4m^4-404l^4m^3+196l^3m^3+1026l^3m^2-350l^2m^2-1080l^2m+264lm+405l-72}{(lm-1)^2(lm-2)^2(lm-3)^2}$
- $TP(G) = \frac{l}{\sqrt{\frac{1}{(lm-1)(lm-2)}}} + \frac{1}{2}\frac{3l+2}{\sqrt{\frac{1}{(lm-2)^2}}} + \frac{1}{6}\frac{l\sqrt{6}}{\sqrt{\frac{1}{(lm-2)(lm-3)}}} + \frac{1}{3}\frac{l-3}{\sqrt{\frac{1}{(lm-3)^2}}}$
- $TRP(G) = \sqrt{6}\sqrt{\frac{1}{(lm-2)(lm-3)}}\,l + \sqrt{\frac{1}{(lm-1)(lm-2)}}\,l + 6\sqrt{\frac{1}{(lm-2)^2}}\,l + 3\sqrt{\frac{1}{(lm-2)^2}}\,l + 9\sqrt{\frac{1}{(lm-2)^2}}$

Proof

1. $TM_1(G) = \sum_{i=1}^{n}(T_u + T_v)$

$TM_1(G) = \left(\frac{1}{lm-1} + \frac{1}{lm-2}\right) l + \left(\frac{2}{lm-2} + \frac{2}{lm-2}\right)(3l+2) + \left(\frac{2}{lm-2} + \frac{3}{lm-3}\right)(l-3)$

$$TM_1(G) = \left(\frac{1}{lm-1} + \frac{1}{lm-2}\right) l + \frac{4(3l+2)}{lm-2} + \left(\frac{2}{lm-2} + \frac{3}{lm-3}\right) l + \frac{6(l-3)}{lm-3}$$

$$M_1(G) = \frac{25l^3m^2 - 10l^2m^2 - 92l^2m + 22lm + 69l - 12}{(lm-1)(lm-2)(lm-3)}$$

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2. $TM_2(G)=\sum_{i=1}^{n}(T_u\times T_v)$

$$TM_2(G)=\left(\frac{1}{lm-1}\cdot\frac{1}{lm-2}\right)l+\left(\frac{2}{lm-2}\cdot\frac{2}{lm-2}\right)(3l+2)+\left(\frac{2}{lm-2}\cdot\frac{3}{lm-3}\right)(l-3)$$

$$TM_2(G)=\frac{l}{(lm-1)(lm-2)}+\frac{4(3l+2)}{(lm-2)^2}+\frac{6l}{(lm-2)(lm-3)}+\frac{9(l-3)}{(lm-3)^2}$$

$$TM_2(G)=\frac{28l^4m^3-19l^3m^3-173l^3m^2+79l^2m^2+339l^2m-96lm-198l+36}{(lm-1)(lm-2)^2(lm-3)^2}$$

3. $TF(G)=\sum_{i=1}^{n}(T_u^2+T_v^2)$

$$TF(G)=\left(\left(\frac{1}{lm-1}\right)^2+\left(\frac{1}{lm-2}\right)^2\right)(l)++\left(\left(\frac{2}{lm-2}\right)^2+\left(\frac{2}{lm-2}\right)^2\right)(3l+2)$$
$$+\left(\left(\frac{2}{lm-2}\right)^2+\left(\frac{3}{lm-3}\right)^2\right)l+\left(\left(\frac{3}{lm-3}\right)^2+\left(\frac{3}{lm-3}\right)^2\right)(l-3)$$

$$TF(G)=\left(\frac{1}{(lm-1)^2}+\frac{1}{(lm-2)^2}\right)l+\frac{8(3l+2)}{(lm-2)^2}+\left(\frac{4}{(lm-2)^2}+\frac{3}{(lm-3)^2}\right)l+\frac{18(l-3)}{(lm-3)^2}$$
$$=\frac{57l^5m^4-38l^4m^4-404l^4m^3+196l^3m^3+1026l^3m^2-350l^2m^2-1080l^2m+264lm+405l-72}{(lm-1)^2(lm-2)^2(lm-3)^2}$$

4. $TP(G)=\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T_uT_v}}\right)$

$$TP(G)=\frac{1}{\sqrt{\left(\frac{1}{lm-1}\cdot\frac{1}{lm-2}\right)}}l+\frac{1}{\sqrt{\left(\frac{2}{lm-1}\cdot\frac{2}{lm-2}\right)}}(3l+2)+\frac{1}{\sqrt{\left(\frac{2}{lm-2}\cdot\frac{3}{lm-3}\right)}}l$$
$$+\frac{1}{\sqrt{\frac{3}{lm-3}\cdot\frac{3}{lm-3}}}(l-3)$$

$$TP(G)=\frac{l}{\sqrt{\frac{1}{(lm-1)(lm-2)}}}+\frac{1}{2}\frac{3l+2}{\sqrt{\frac{1}{(lm-2)^2}}}+\frac{1}{6}\frac{l\sqrt{6}}{\sqrt{\frac{1}{(lm-2)(lm-3)}}}+\frac{1}{3}\frac{l-3}{\sqrt{\frac{1}{(lm-3)^2}}}$$

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5. $TRP(G) = \sum_{i=1}^{n}\left(\sqrt{T_u T_v}\right)$

$$TRP(G) = \sqrt{\left(\frac{1}{lm-1}\cdot\frac{1}{lm-2}\right)}\,l + \sqrt{\left(\frac{2}{lm-1}\cdot\frac{2}{lm-2}\right)}\,(3l+2) + \sqrt{\left(\frac{2}{lm-2}\cdot\frac{3}{lm-3}\right)}\,l + \sqrt{\frac{3}{lm-3}\cdot\frac{3}{lm-3}}\,(l-3)$$

$$TRP(G) = \sqrt{\frac{1}{(lm-1)(lm-2)}}\,l + 2\sqrt{\frac{1}{(lm-2)^2}}\,(3l+2) + \sqrt{6}\sqrt{\frac{1}{(lm-2)(lm-3)}}\,l + 3\sqrt{\frac{1}{(lm-3)^2}}\,(l-3)$$

$$\mathrm{T}RP = \sqrt{6}\sqrt{\frac{1}{(lm-2)(lm-3)}}\,l + \sqrt{\frac{1}{(lm-1)(lm-2)}}\,l + 6\sqrt{\frac{1}{(lm-2)^2}}\,l + 3\sqrt{\frac{1}{(lm-2)^2}}\,l + 9\sqrt{\frac{1}{(lm-2)^2}}$$

2.2 Bridge graph over cycle 1. Bridge graph over path $G_l(C_m, v)$ over C_m

Assuming V is the arrangement of vertices saw in Figure 2, this arrangement of vertices can be parted into four subsets, such that $V = V_1 + V_2 + V_3 + V_4$. Figure 2 shows a molecular graph of bridge graph over cycle. Table 2 provides a detailed description of the edge partition of bridge graph over cycle [11].

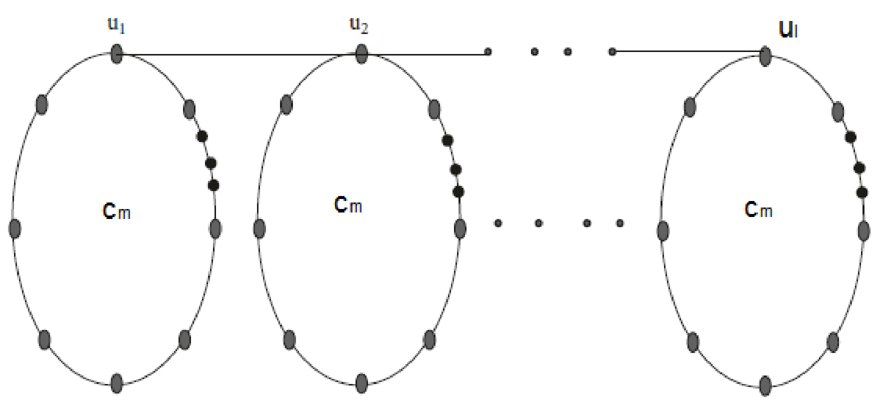

Figure 2: Graph of Bridge graph over cylce

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d_u, d_v	$\frac{d_u}{n-d_u}$	$\frac{d_v}{n-d_v}$	*Recurrence*
(2,2)	$\frac{2}{lm-1}$	$\frac{2}{lm-2}$	$lm-2l$
(2,3)	$\frac{2}{lm-3}$	$\frac{3}{lm-3}$	4
(2,4)	$\frac{2}{lm-2}$	$\frac{4}{lm-3}$	$2l-4$
(3,4)	$\frac{3}{lm-3}$	$\frac{4}{lm-4}$	2
(4,4)	$\frac{4}{lm-4}$	$\frac{4}{lm-4}$	$l-3$

Table 2: Edge partition table of bridge graph over cycle

Theorem 2

Let G be a graph of $\boldsymbol{G_l(C_m, v)}$ ***over*** $\boldsymbol{C_m}$ ***then its temperature – based indices are***

- $TM_1(G)=\frac{3l^3m^2-14l^3m^2-35l^2m^2-66l^2m+88lm+72l-48}{(lm-2)(lm-3)(lm-4)}$
- $TM_2(G)=\frac{28l^4m^3-19l^3m^3-173l^3m^2+79l^2m^2+339l^2m-96lm-198l+36}{(lm-1)(lm-2)^2(lm-3)^2}$
- $TF(G)=\frac{5l^5m^5+62l^5m^4-144l^4m^4-612l^4m^3+997l^3m^3+222l^3m^2-2768l^2m^2-350l^2m+3216lm+2016l-1152}{(lm-2)^2(lm-3)^2(lm-4)^2}$
- $TP(G)=\frac{1}{2}\frac{(lm-2l)\sqrt{2}}{\sqrt{\frac{1}{(lm-2)^2}}}+\frac{2}{3}\frac{\sqrt{6}}{\sqrt{\frac{1}{(lm-2)(lm-3)}}}+\frac{1}{4}\frac{(2l-4)\sqrt{2}}{\sqrt{\frac{1}{(lm-2)(lm-3)}}}+\frac{1}{3}\frac{\sqrt{3}}{\sqrt{\frac{1}{(lm-3)(lm-4)}}}+\frac{1}{4}\frac{l-3}{\sqrt{\frac{1}{(lm-4)^2}}}$
- $TRP(G)=\sqrt{2}\sqrt{\frac{1}{(lm-2)^2}}\,lm-2\sqrt{2}\sqrt{\frac{1}{(lm-2)^2}}\,l+4\sqrt{2}\sqrt{\frac{1}{(lm-2)(lm-4)}}\,l-$
  $8\sqrt{2}\sqrt{\frac{1}{(lm-2)(lm-4)}}+$
- $4\sqrt{6}\sqrt{\frac{1}{(lm-2)(lm-3)}}+4\sqrt{3}\sqrt{\frac{1}{(lm-3)(lm-4)}}+4\sqrt{\frac{1}{(lm-4)^2}}\,l-12\sqrt{\frac{1}{(lm-4)^2}}\,l$

Proof

1. $TM_1(G)=\sum_{i=1}^{n}(T_u+T_v)$

$$TM_1(G)=\left(\frac{2}{lm-2}+\frac{1}{lm-2}\right)(lm-2l)+\left(\frac{2}{lm-2}+\frac{3}{lm-3}\right)4+\left(\frac{2}{lm-2}+\frac{4}{lm-4}\right)(2l-4)+\left(\frac{3}{lm-3}+\frac{4}{lm-4}\right)2+\left(\frac{4}{lm-4}+\frac{4}{lm-4}\right)(l-3)$$

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$$=\frac{3(lm-2l)}{lm-2}+\frac{8}{lm-2}+\frac{18}{lm-3}+\left(\frac{2}{lm-2}+\frac{4}{lm-4}\right)(2l-4)+\frac{8}{lm-4}+\frac{8(l-3)}{lm-4}$$

$$TM_1(G)=\frac{3l^3m^2-14l^3m^2-35l^2m^2-66l^2m+88lm+72l-48}{(lm-2)(lm-3)(lm-4)}$$

2. $TM_2(G)=\sum_{i=1}^{n}(T_u\times T_v)$

$$TM_2(G)=\left(\frac{2}{lm-2}\cdot\frac{1}{lm-2}\right)(lm-2l)+\left(\frac{2}{lm-2}\cdot\frac{3}{lm-3}\right)(4)+\left(\frac{2}{lm-2}\cdot\frac{4}{lm-4}\right)(2l-4)+\left(\frac{4}{lm-4}\cdot\frac{4}{lm-4}\right)(l-3)$$

$$=\frac{2(lm-2l)}{(lm-2)^2}+\frac{24}{(lm-2)(lm-3)}+\frac{8(2l-4)}{(lm-2)(lm-4)}+\frac{24}{(lm-3)(lm-4)}+\frac{16(l-3)}{(lm-4)^2}$$

$$TM_2(G)=\frac{2(l^3m^3+14l^3m^2-24l^2m^2-64l^2m+64lm+64l-32)}{(lm-4)^2(lm-2)^2}$$

$$TM_2(G)=\frac{28l^4m^3-19l^3m^3-173l^3m^2+79l^2m^2+339l^2m-96lm-198l+36}{(lm-1)(lm-2)^2(lm-3)^2}$$

3. $TF(G)=\sum_{i=1}^{n}(T_u^2+T_v^2)$

$$=\left(\left(\frac{2}{lm-2}\right)^2+\left(\frac{1}{lm-2}\right)^2\right)(lm-2l)+\left(\left(\frac{2}{lm-2}\right)^2+\left(\frac{2}{lm-2}\right)^2\right)(4)+\left(\left(\frac{2}{lm-2}\right)^2+\left(\frac{4}{lm-4}\right)^2\right)(2l-4)+\left(\left(\frac{3}{lm-3}\right)^2+\left(\frac{4}{lm-4}\right)^2\right)(2)+\left(\left(\frac{4}{lm-4}\right)^2+\left(\frac{4}{lm-4}\right)^2\right)(l-3)$$

$$=\frac{5(lm-2l)}{(lm-1)^2}+\frac{16}{(lm-2)^2}+\frac{16}{(lm-2)^2}+\frac{54}{(lm-3)^2}\left(\frac{4}{(lm-2)^2}+\frac{16}{(lm-4)^2}\right)(2l-4)+\frac{32}{(lm-4)^2}+\frac{32(l-3)}{(lm-4)^2}$$

TF (G)= $\frac{5l^5m^5+62l^5m^4-144l^4m^4-612l^4m^3+997l^3m^3+222l^3m^2-2768l^2m^2-350l^2m+3216lm+2016l-1152}{(lm-2)^2(lm-3)^2(lm-4)^2}$

4. $TP(G)=\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T_uT_v}}\right)$

$$=\frac{1}{\sqrt{\left(\frac{2}{lm-1}\cdot\frac{1}{lm-2}\right)}}(lm-2l)+\frac{1}{\sqrt{\left(\frac{2}{lm-1}\cdot\frac{3}{lm-3}\right)}}(4)+\frac{1}{\sqrt{\left(\frac{2}{lm-2}\cdot\frac{4}{lm-4}\right)}}(2l-4)+\frac{1}{\sqrt{\frac{3}{lm-3}\cdot\frac{4}{lm-4}}}(2)+\frac{1}{\sqrt{\frac{4}{lm-3}\cdot\frac{4}{lm-4}}}(l-3)$$

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$$= \frac{1}{2}\frac{(lm-2l)\sqrt{2}}{\sqrt{\frac{1}{(lm-2)^2}}} + \frac{2}{3}\frac{\sqrt{6}}{\sqrt{\frac{1}{(lm-2)(lm-3)}}} + \frac{1}{4}\frac{(2l-4)\sqrt{2}}{\sqrt{\frac{1}{(lm-2)(lm-3)}}} + \frac{1}{3}\frac{\sqrt{3}}{\sqrt{\frac{1}{(lm-3)(lm-4)}}} + \frac{1}{4}\frac{l-3}{\sqrt{\frac{1}{(lm-4)^2}}}$$

5. $TRP(G) = \sum_{i=1}^{n}\left(\sqrt{T_u T_v}\right)$

$$TRP(G) = \sqrt{\left(\frac{2}{lm-2}\cdot\frac{1}{lm-2}\right)}(lm-2l) + \sqrt{\left(\frac{2}{lm-2}\cdot\frac{3}{lm-3}\right)}(4) + \sqrt{\left(\frac{2}{lm-2}\cdot\frac{4}{lm-4}\right)}(2l-4) + \sqrt{\frac{3}{lm-3}\cdot\frac{4}{lm-4}}(2) + \sqrt{\frac{4}{lm-4}\cdot\frac{4}{lm-4}}(l-3)$$

$$TRP(G) = \sqrt{2}\sqrt{\frac{1}{(lm-2)^2}}(lm-2l) + 4\sqrt{6}\sqrt{\frac{1}{(lm-2)(lm-3)}} + 2\sqrt{2}\sqrt{\frac{1}{(lm-2)(lm-4)}}(2l-4) + 4\sqrt{3}\sqrt{\frac{1}{(lm-3)(lm-4)}} + (4)\sqrt{\frac{1}{(lm-4)^2}}(l-3)$$

$$TRP(G) = \sqrt{2}\sqrt{\frac{1}{(lm-2)^2}}\,lm - 2\sqrt{2}\sqrt{\frac{1}{(lm-2)^2}}\,l + 4\sqrt{2}\sqrt{\frac{1}{(lm-2)(lm-4)}}\,l - 8\sqrt{2}\sqrt{\frac{1}{(lm-2)(lm-4)}} + 4\sqrt{6}\sqrt{\frac{1}{(lm-2)(lm-3)}} + 4\sqrt{3}\sqrt{\frac{1}{(lm-3)(lm-4)}} + 4\sqrt{\frac{1}{(lm-4)^2}}\,l - 12\sqrt{\frac{1}{(lm-4)^2}}\,l$$

2.3 Bridge graph over cycle $G_l(k_m, v)$ over k_m, $m > 2$

Assuming that vertices set is V, understanding Figure 3 allows us to divide set of vertices into three subsets V_1, V_2, and V_3 so that $V = V_1 + V_2 + V_3$. If E shows the edge set. Figure 6 shows the bridge graph G_l (K_m, v) of the complete graph of the hybrid network. Bridge

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Graph of the network graph has five different edges. Table 3 provides a detailed description of the edge set. [11]

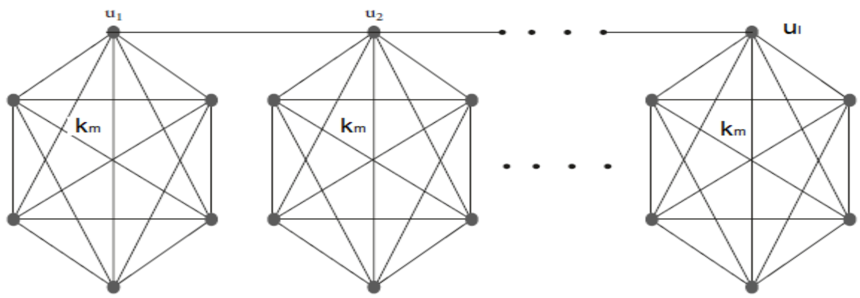

Figure 3: Bridge graph over complete graph

d_u, d_v	$\frac{d_u}{n-d_u}$	$\frac{d_v}{n-d_v}$	*Recurrence*
$(4,5)$	$\frac{4}{lm-4}$	$\frac{5}{lm-4}$	2
$(4, m-1)$	$\frac{4}{lm-4}$	$\frac{m-1}{lm-m+1}$	2
$(5,5)$	$\frac{5}{lm-5}$	$\frac{5}{lm-5}$	$l-2$
$(5, m-1)$	$\frac{5}{lm-5}$	$\frac{m-1}{lm-m+1}$	$l-2$
$(m-1, m-1)$	$\frac{m-1}{lm-m+1}$	$\frac{m-1}{lm-m+1}$	$\frac{lm(l-1)-2(l+1)}{2}$

Table 3: Edge partition of Bridge graph over complete graph

Theorem 3

Let G be a graph of $\boldsymbol{G_l(k_m, v)}$ over $\boldsymbol{k_m}$for$\boldsymbol{m > 2}$ then its temperature –based indices are

- $TM_1(G) = \frac{l^4m^4-l^4m^3-l^3m^4-9l^3m^3+25l^3m^2+7l^2m^3+3l^2m^2-74l^2m+2lm^2+38lm-40l-40m+40}{(lm-4)(lm-5)(lm-m+1)}$
- $TM_2(G) = \frac{2(l^3m^3+14l^3m^2-24l^2m^2-64l^2m+64lm+64l-32)}{(lm-4)^2(lm-2)^2}$
- TF(G)=$\frac{1}{(lm-4)^2(lm-5)(lm-m+1)^2}(l^6m^7 - 2l^6m^6 - l^5m^7 + l^6m^5 - 17l^5m^6 + 37l^5m^5 + 16l^4m^6 + 56l^5m^4 - 107l^4m^5) - 448l^4m^4 + 85l^3m^5 + 311l^4m^3 - 164l^3m^4 + 2015l^3m^3 + 82l^2m^4 - 406l^3m^2 - 324l^2m^3 - 2318l^2m^2 + 480lm^3 2560l^2m - 160lm^2 - 1120lm - 800m^2 + 800l + 1600m - 800$

Jilin Daxue Xuebao (Gongxueban)/Journal of Jilin University (Engineering and Technology Edition)
ISSN: 1671-5497
E-Publication: Online Open Access
Vol: 42 Issue: 06-2023
DOI 10.17605/OSF.IO/X483N

- $TP(G)\frac{1}{5}\frac{\sqrt{5}}{\sqrt{\frac{1}{(lm-4)(lm-5)}}}+\frac{1}{\sqrt{\frac{m-1}{(lm-4)(lm-m+1)}}}+\frac{1}{5}\frac{l-2}{\sqrt{\frac{1}{(lm-5)^2}}}+\frac{1}{5}\frac{\sqrt{5}(l-2)}{\sqrt{\frac{m-1}{(lm-5)(lm-m+1)}}}+\frac{1}{2}\frac{lm(l-1)-2l-2}{\sqrt{\frac{(m-1)^2}{(lm-m+1)^2}}}$
- TRP(G)=$\sqrt{5}\sqrt{\frac{1}{(lm-4)(lm-5)}}+4\sqrt{\frac{m-1}{(lm-4)(lm-m+1)}}+5\sqrt{\frac{1}{(lm-5)^2}}(l-2)+\sqrt{5}\sqrt{\frac{m-1}{(lm-5)(lm-m+1)}}(l-2)\sqrt{\frac{(m-1)^2}{(lm-m+1)^2}}(\frac{1}{2}lm(l-1)-l-1)$

Proof

1. $TM_1(G)=\sum_{i=1}^{n}(T_u+T_v)$

$$TM_1(G)=\left(\frac{4}{lm-4}+\frac{5}{lm-5}\right)(2)+\left(\frac{4}{lm-4}+\frac{m-1}{lm-m+1}\right)2+\left(\frac{5}{lm-5}+\frac{5}{lm-5}\right)(l-2)+\left(\frac{5}{lm-5}+\frac{m-1}{lm-m+1}\right)(l-2)+\left(\frac{m-1}{lm-m+1}+\frac{m-1}{lm-m+1}\right)\left(\frac{lm(l-1)-2(l+1)}{2}\right)$$

$$TM_1(G)=\frac{16}{lm-4}+\frac{10}{lm-5}+\frac{2(m-1)}{lm-m+1}+\frac{10(l-2)}{lm-5}+\left(\frac{5}{lm-5}+\frac{m-1}{lm-m+1}\right)(l-2)+\frac{2(m-1)\left(\frac{1}{2}lm(l-1)-l-1\right)}{lm-m+1}$$

$$=\frac{l^4m^4-l^4m^3-l^3m^4-9l^3m^3+25l^3m^2+7l^2m^3+3l^2m^2-74l^2m+2lm^2+38lm-40l-40m+40}{(lm-4)(lm-5)(lm-m+1)}$$

2. $TM_2(G)=\sum_{i=1}^{n}(T_u\times T_v)$

$$=\left(\frac{4}{lm-4}\cdot\frac{5}{lm-5}\right)(2)+\left(\frac{4}{lm-4}\cdot\frac{m-1}{lm-m+1}\right)2+\left(\frac{5}{lm-5}\cdot\frac{5}{lm-5}\right)(l-2)+\left(\frac{5}{lm-5}\cdot\frac{m-1}{lm-m+1}\right)(l-2)+\left(\frac{m-1}{lm-m+1}\cdot\frac{m-1}{lm-m+1}\right)$$

$$=\frac{40}{(lm-4)(lm-5)}+\frac{8(m-1)}{(lm-4)(lm-m+1)}+\frac{25(l-2)}{(lm-5)^2}+\frac{5(m-1)(l-2)}{(lm-5)(lm-m+1)}+\frac{(m-1)^2\left(\frac{1}{2}lm(l-1)-l-1\right)}{(lm-m+1)^2}+\frac{24}{(lm-3)(lm-4)}+\frac{16(l-3)}{(lm-4)^2}$$

Jilin Daxue Xuebao (Gongxueban)/Journal of Jilin University (Engineering and Technology Edition)
ISSN: 1671-5497
E-Publication: Online Open Access
Vol: 42 Issue: 06-2023
DOI 10.17605/OSF.IO/X483N

$$= \frac{1}{(lm-5)^2(lm-4)(lm-m+1)^2}(l^5m^6 - 2l^5m^5 - l^5m^4 - 14l^4m^5 + 41l^4m^4 + 12l^3m^5 + 24l^4m^3 + 55l^3m^4 - 360l^3m^3) - 33l^2m^4 + 73l^3m^2 + 36l^2m^3 + 687l^2m^2 - 70lm^3 - 690l^2m - 60lm^2 + 330lm + 200m^2 - 200l - 400m + 200$$

$$TM_2(G) = \frac{2(l^3m^3 + 14l^3m^2 - 24l^2m^2 - 64l^2m + 64lm + 64l - 32)}{(lm-4)^2(lm-2)^2}$$

3. $TF(G) = \sum_{i=1}^{n}(T_u^2 + T_v^2)$

TF (G)= $\left(\left(\frac{4}{lm-4}\right)^2 + \left(\frac{5}{lm-5}\right)^2\right)(2) + \left(\left(\frac{4}{lm-4}\right)^2 + \left(\frac{m-1}{lm-m+1}\right)^2\right)(2) + \left(\left(\frac{5}{lm-5}\right)^2 + \left(\frac{5}{lm-5}\right)^2\right)(l-2) + \left(\left(\frac{5}{lm-5}\right)^2 + \left(\frac{m-1}{lm-m+1}\right)^2\right)(l-2) + \left(\left(\frac{m-1}{lm-m+1}\right)^2 + \left(\frac{m-1}{lm-m+1}\right)^2\right)\frac{lm(l-1)-2(l+1)}{2}$

$= \frac{64}{(lm-4)^2} + \frac{50}{(lm-5)^2} + \frac{2(m-1)^2}{(lm-m+1)^2} + \frac{50(l-2)}{(lm-5)^2} + \left(\frac{25}{(lm-5)^2} + \frac{(m-1)^2}{(lm-m+1)^2}\right)(l-2) + \frac{2(m-1)^2\left(\frac{1}{2}lm(l-1)-2(l+1)\right)}{(lm-m+1)^2}$

$=\frac{1}{(lm-4)^2(lm-5)(lm-m+1)^2}(l^6m^7 - 2l^6m^6 - l^5m^7 + l^6m^5 - 17l^5m^6 + 37l^5m^5 + 16l^4m^6 + 56l^5m^4 - 107l^4m^5) - 448l^4m^4 + 85l^3m^5 + 311l^4m^3 - 164l^3m^4 + 2015l^3m^3 + 82l^2m^4 - 406l^3m^2 - 324l^2m^3 - 2318l^2m^2 + 480lm^3 2560l^2m - 160lm^2 - 1120lm - 800m^2 + 800l + 1600m - 800$

4. $TP(G) = \sum_{i=1}^{n}\left(\frac{1}{\sqrt{T_uT_v}}\right)$

$$= \frac{1}{\sqrt{\left(\frac{4}{lm-4}\cdot\frac{5}{lm-5}\right)}}(2) + \frac{1}{\sqrt{\left(\frac{4}{lm-4}\cdot\frac{m-1}{lm-m+1}\right)}}(2) + \frac{1}{\sqrt{\left(\frac{5}{lm-5}\cdot\frac{m-1}{lm-m+1}\right)}}(l-2) + \frac{1}{\sqrt{\frac{m-1}{lm-m+1}\cdot\frac{m-1}{lm-m+1}}}\left(\frac{lm(l-1)-2(l+1)}{2}\right)$$

$$= \frac{1}{5}\frac{\sqrt{5}}{\sqrt{\frac{1}{(lm-4)(lm-5)}}} + \frac{1}{\sqrt{\frac{m-1}{(lm-4)(lm-m+1)}}} + \frac{1}{5}\frac{l-2}{\sqrt{\frac{1}{(lm-5)^2}}} + \frac{1}{5}\frac{\sqrt{5}(l-2)}{\sqrt{\frac{m-1}{(lm-5)(lm-m+1)}}} + \frac{1}{2}\frac{lm(l-1)-2l-2}{\sqrt{\frac{(m-1)^2}{(lm-m+1)^2}}}$$

Jilin Daxue Xuebao (Gongxueban)/Journal of Jilin University (Engineering and Technology Edition)
ISSN: 1671-5497
E-Publication: Online Open Access
Vol: 42 Issue: 06-2023
DOI 10.17605/OSF.IO/X483N

5. $TRP(G) = \sum_{i=1}^{n}\left(\sqrt{T_u T_v}\right)$

$$= \sqrt{\left(\frac{4}{lm-4}\cdot\frac{5}{lm-5}\right)}(2) + \sqrt{\left(\frac{4}{lm-4}\cdot\frac{m-1}{lm-m+1}\right)}(2) + \sqrt{\left(\frac{5}{lm-5}\cdot\frac{5}{lm-5}\right)}(l-2)$$

$$+ \sqrt{\frac{5}{lm-5}\cdot\frac{m-1}{lm-m+1}}(l-2)$$

$$+ \sqrt{\frac{m-1}{lm-m+1}\cdot\frac{m-1}{lm-m+1}}\left(\frac{lm(l-1)-2l-2}{2}\right)$$

$$=\sqrt{5}\sqrt{\frac{1}{(lm-4)(lm-5)}} + 4\sqrt{\frac{m-1}{(lm-4)(lm-m+1)}} + 5\sqrt{\frac{1}{(lm-5)^2}}(l-2) + \sqrt{5}\sqrt{\frac{m-1}{(lm-5)(lm-m+1)}}(l-2) + \sqrt{\frac{(m-1)^2}{(lm-m+1)^2}}\left(\frac{1}{2}lm(l-1)-l-1\right)$$

Graphical Comparison of Results:

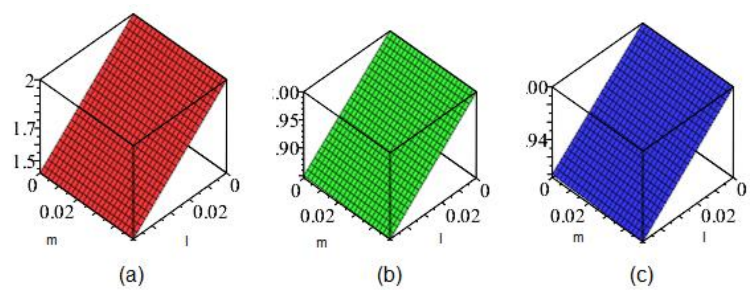

Figure 4: Temperature based First Zagreb index for Bridge graph over path (a), cycle (b) and complete (c) graphs

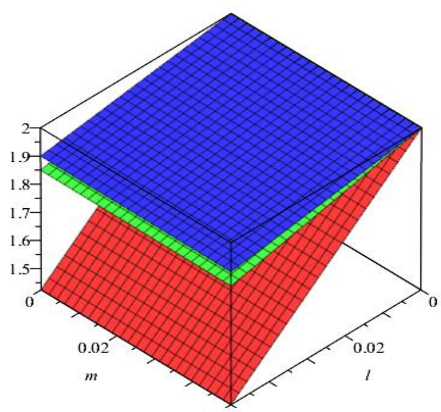

Figure 5: Comparison of Temperature based First Zagreb index for Bridge graph over path, cycle and complete graph

Jilin Daxue Xuebao (Gongxueban)/Journal of Jilin University (Engineering and Technology Edition)
ISSN: 1671-5497
E-Publication: Online Open Access
Vol: 42 Issue: 06-2023
DOI 10.17605/OSF.IO/X483N

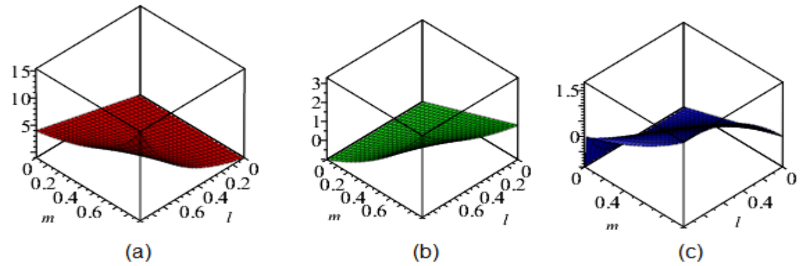

Figure 6: Temperature based Second Zagreb index for Bridge graph over path (a), cycle (b) and complete (c) graphs

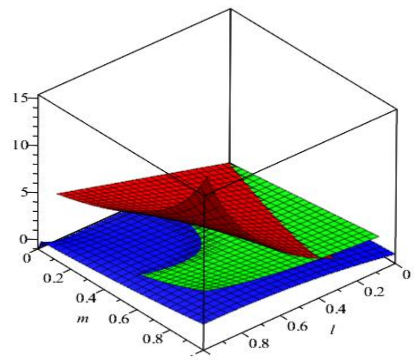

Figure 7: Comparison of Temperature based Second Zagreb index for Bridge graph over path, cycle and complete graph.

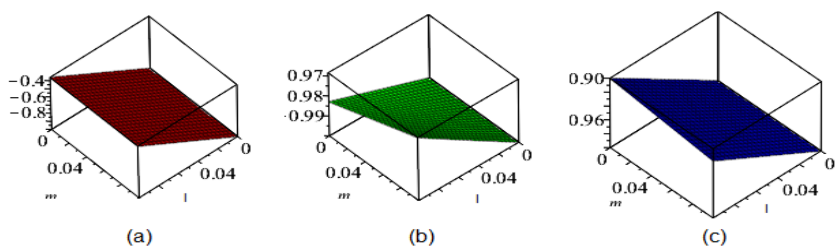

Figure 8: Temperature based forgotten index for Bridge graph over path (a), cycle (b) and complete (c) graphs

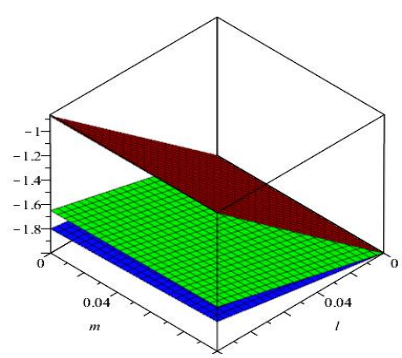

Figure 9: Comparison of Temperature based forgotten index for Bridge graph over path, cycle and complete graphs

Jilin Daxue Xuebao (Gongxueban)/Journal of Jilin University (Engineering and Technology Edition)
ISSN: 1671-5497
E-Publication: Online Open Access
Vol: 42 Issue: 06-2023
DOI 10.17605/OSF.IO/X483N

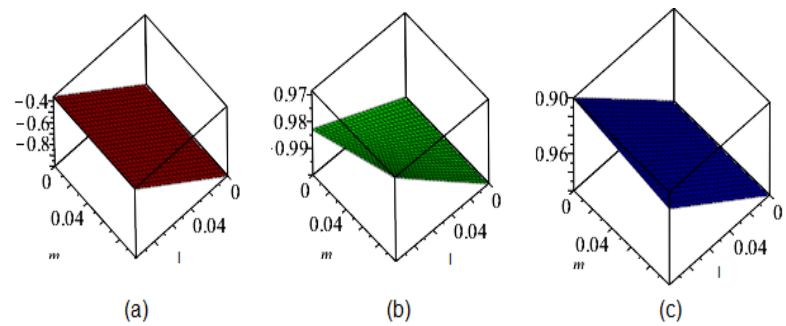

Figure 10: Temperature based Productivity index for Bridge graph over path, cycle and complete graphs

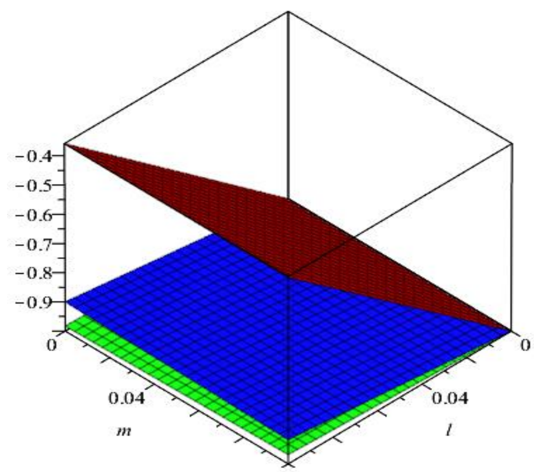

Figure 11: Comparison of Temperature based productivity index for Bridge graph over path, cycle and complete graphs

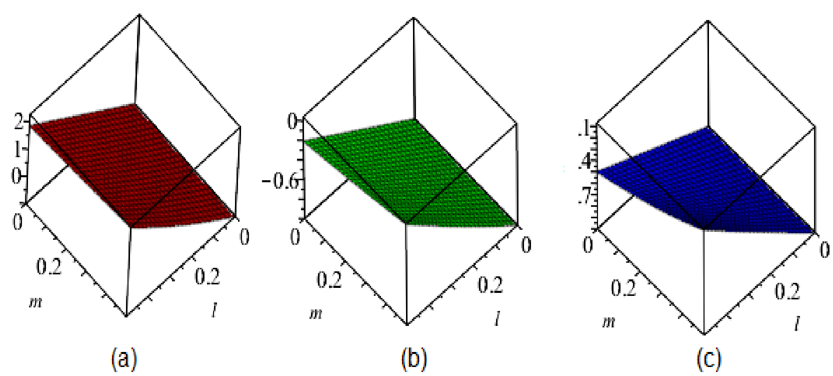

Figure 12: Temperature based reciprocal Productivity index for Bridge graph over path, cycle and complete graphs

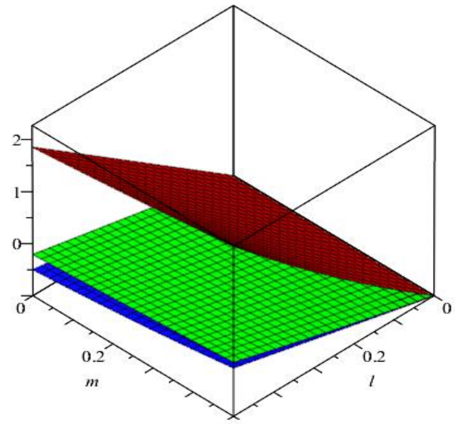

Figure 13: Comparison of Temperature based reciprocal productivity index for Bridge graph over path, cycle and complete graphs

3. CONCLUSIONS

This paper focused on the study of specific type of topological indices namely temperature based indices for bridge graphs. Bridge graphs are a specific type of graph where every edge is a bridge, meaning that its removal would disconnect the graph. The investigation of topological indices for bridge graphs is important as it provides insights into the structural properties and characteristics of these graphs. The paper reviewed the temperature based version of several topological indices commonly used in graph theory, namely 1st and 2nd Zagreb indices, forgotten topological index, productivity index and reciprocal productivity index. It explored the applicability of these indices to bridge graph over path, cycle and complete graph also examined their behavior and properties when applied to this specific class of graphs. Graphical comparisons of our numerical results are also included in this paper.

Data Availability

The data used in this paper can be requested from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this work.

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ISSN: 1671-5497
E-Publication: Online Open Access
Vol: 42 Issue: 06-2023
DOI 10.17605/OSF.IO/X483N

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